Chapter - Integrals

Topic-1: Standard Integrals, Integration by Substitution, Itegration by Parts

1 MCQs with One Correct Answer

1. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}} dx$ equals (for some arbitrary constant K) [2012]

(a) $-\frac{1}{(\sec x + \tan x)^{\frac{11}{2}}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(b) $\frac{1}{(\sec x + \tan x)^{\frac{11}{2}}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(c) $-\frac{1}{(\sec x + \tan x)^{\frac{11}{2}}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(d) $\frac{1}{(\sec x + \tan x)^{\frac{11}{2}}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

2. Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx$. Then, for an arbitrary constant C, the value of J – I equals [2008]

(a) $\frac{1}{2}\log\left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right) + C$

(b) $\frac{1}{2}\log\left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1}\right) + C$

(c) $\frac{1}{2}\log\left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right) + C$

(d) $\frac{1}{2}\log\left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right) + C$

3. The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$ is [1995S]

(a) $\sin x - 6\tan^{-1}(\sin x) + c$

(b) $\sin x - 2(\sin x)^{-1} + c$

(c) $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + c$

(d) $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x) + c$

4 Fill in the Blanks

4. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B\log(9e^{2x} - 4) + C$, then $A =$ ...., $B =$ .... and $C =$ .... [1990 - 2 Marks]

6 MCQs with One or More than One Correct Answer

5. Let b be a nonzero real number. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f(0) = 1$. If the derivative f' of f satisfies the equation

$$f'(x) = \frac{f(x)}{b^2 + x^2}$$

for all $x \in \mathbb{R}$, then which of the following statements is/are TRUE? [Adv. 2020]

(a) If $b > 0$, then f is an increasing function

(b) If $b < 0$, then f is a decreasing function

(c) $f(x)f(-x) = 1$ for all $x \in \mathbb{R}$

(d) $f(x) - f(-x) = 0$ for all $x \in \mathbb{R}$

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions satisfying $f(x+y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$ for all $x, y \in \mathbb{R}$. If $\lim_{x\to 0} g(x) = 1$, then which of the following statements is/are TRUE? [Adv. 2020]

(a) f is differentiable at every $x \in \mathbb{R}$

(b) If $g(0) = 1$, then g is differentiable at every $x \in \mathbb{R}$

(c) The derivative $f'(1)$ is equal to 1
(d) The derivative $f'(0)$ is equal to 1

9 Assertion and Reason/Statement Type Questions

7. Let $F(x)$ be an indefinite integral of $\sin^2 x$.
STATEMENT-1 : The function $F(x)$ satisfies $F(x+\pi) = F(x)$ for all real x. because
STATEMENT-2 : $\sin^2(x+\pi) = \sin^2 x$ for all real x. [2007 -3 marks]
(a) Statement-1 is True, statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True.

10 Subjective Problems

8. For any natural number m, evaluate
$$\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}dx, x>0.$$ [2002 - 5 Marks]

9. Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$. [2001 - 5 Marks]

10. Evaluate $\int \frac{(x+1)}{x(1+xe^x)^2}dx$. [1996 - 2 Marks]

11. Find the indefinite integral $\int \cos 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$ [1994 - 5 Marks]

12. Find the indefinite integral $\int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{4}}+\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$ [1992 - 4 Marks]

13. Evaluate $\int(\sqrt{\tan x}+\sqrt{\cot x})dx$ [1989 - 3 Marks]

14. Evaluate : $\int\left[\frac{(\cos 2x)^{1/2}}{\sin x}\right]dx$ [1987 - 6 Marks]

15. Evaluate the following $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$ [1985 - 2½ Marks]

16. Evaluate the following $\int\frac{dx}{x^2(x^4+1)^{3/4}}$ [1984 - 2 Marks]

17. Evaluate $\int(e^{\log x}+\sin x)\cos x\,dx$. [1981 - 2 Marks]

18. Evaluate $\int\frac{x^2dx}{(a+bx)^2}$ [1979]

19. Evaluate $\int\frac{\sin x}{\sin x-\cos x}dx$ [1978]

Topic-2: Integration of the Forms: $\int e^x(f(x)+f'(x))dx$, $\int e^{kx}(kf(x)+f'(x))dx$, Integration by Partial Fractions, Integration of Different Expression of e^x

1 MCQs with One Correct Answer

1. $\int\frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx =$ [2006 - 3M, –1]
(a) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+c$ (b) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+c$
(c) $\frac{\sqrt{2x^4-2x^2+1}}{x}+c$ (d) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

10 Subjective Problems

2. Integrate : $\int\frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx.$ [1999 - 5 Marks]

3. Evaluate : $\int\frac{(x-1)e^x}{(x+1)^3}dx$ [1983 - 2 Marks]

Topic-3: Evaluation of Definite Integral by Substitution, Properties of Definite Integrals

1 MCQs with One Correct Answer

1. Let $f:(0,1)\to\mathbb{R}$ be the function defined as $f(x)=\sqrt{n}$ if $x\in\left[\frac{1}{n+1},\frac{1}{n}\right)$ where $n\in\mathbb{N}$. Let $g:(0,1)\to\mathbb{R}$ be a function such that $\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt < g(x) < 2\sqrt{x}$ for all $x\in(0,1)$. Then $\lim_{x\to 0} f(x)g(x)$ [Adv. 2023]
(a) does NOT exist (b) is equal to 1
(c) is equal to 2 (d) is equal to 3

2. For positive integer n, define

$$f(n)=n+\frac{16+5n-3n^2}{4n+3n^2}+\frac{32+n-3n^2}{8n+3n^2}+\frac{48-3n-3n^2}{12n+3n^2}+\ldots+\frac{25n-7n^2}{7n^2}.$$

Then, the value of $\lim_{n\to\infty} f(n)$ is equal to [Adv. 2022]

(a) $3+\frac{4}{3}\log_e 7$ (b) $4-\frac{3}{4}\log_e\left(\frac{7}{3}\right)$

(c) $4-\frac{4}{3}\log_e\left(\frac{7}{3}\right)$ (d) $3+\frac{3}{4}\log_e 7$

3. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+e^x}dx$ is equal to [Adv. 2016]

(a) $\frac{\pi^2}{4}-2$ (b) $\frac{\pi^2}{4}+2$ (c) $\pi^2-e^{\frac{\pi}{2}}$ (d) $\pi^2+e^{\frac{\pi}{2}}$

4. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\operatorname{cosec} x)^{17}\,dx$ is equal to [Adv. 2014]

(a) $\int_0^{\log(1+\sqrt{2})} 2\left(e^u+e^{-u}\right)^{16} du$

(b) $\int_0^{\log(1+\sqrt{2})} \left(e^u+e^{-u}\right)^{17} du$

(c) $\int_0^{\log(1+\sqrt{2})} \left(e^u-e^{-u}\right)^{17} du$

(d) $\int_0^{\log(1+\sqrt{2})} 2\left(e^u-e^{-u}\right)^{16} du$

5. Let $f:[0,2]\to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let $F(x)=\int_0^{x^2} f(\sqrt{t})\,dt$ for $x\in[0,2]$. If $F'(x)=f'(x)$ for all $x\in(0,2)$, then $F(2)$ equals [Adv. 2014]

(a) e^2-1 (b) e^4-1

(c) $e-1$ (d) e^4

6. Let $f:\left[\frac{1}{2},1\right]\to R$ (the set of all real number) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right)=1$. Then the value of $\int_{1/2}^{1} f(x)\,dx$ lies in the interval [Adv. 2013]

(a) $(2e-1, 2e)$ (b) $(e-1, 2e-1)$

(c) $\left(\frac{e-1}{2}, e-1\right)$ (d) $\left(0,\frac{e-1}{2}\right)$

7. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is [2012]

(a) 0 (b) $\frac{\pi^2}{2}-4$ (c) $\frac{\pi^2}{2}+4$ (d) $\frac{\pi^2}{2}$

8. The value of $\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}} \frac{x\sin x^2}{\sin x^2+\sin(\ell n6-x^2)}dx$ is [2011]

(a) $\frac{1}{4}\ell n\frac{3}{2}$ (b) $\frac{1}{2}\ell n\frac{3}{2}$ (c) $\ell n\frac{3}{2}$ (d) $\frac{1}{6}\ell n\frac{3}{2}$

9. Let f be a real-valued function defined on the interval $(-1, 1)$ such that $e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}\,dt$, for all $x\in(-1,1)$, and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to [2010]

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{e}$

10. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x\sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)\,dt,\quad 0\le x\le 1,$ and $f(0)=0$, then [2009]

(a) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(b) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(c) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

(d) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

11. $\int_{-2}^{0}\{x^3+3x^2+3x+3+(x+1)\cos(x+1)\}\,dx$ is equal to [2005S]

(a) -4 (b) 0 (c) 4 (d) 6

12. If $\int_{\sin x}^{1} t^2 f(t)\,dt=1-\sin x$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is [2005S]

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) 3 (d) $\sqrt{3}$

13. The value of the integral $\int_0^1\sqrt{\frac{1-x}{1+x}}dx$ is [2004S]

(a) $\frac{\pi}{2}+1$ (b) $\frac{\pi}{2}-1$ (c) -1 (d) 1

14. If $f(x)$ is differentiable and $\int_0^{t^2} xf(x)dx = \frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ equals [2004S]
(a) 2/5 (b) –5/2 (c) 1 (d) 5/2

15. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in [2003S]
(a) $(-2, 2)$ (b) no value of x
(c) $(0, \infty)$ (d) $(-\infty, 0)$

16. The integral $\int_{-1/2}^{1/2}\left([x] + \ell n\left(\frac{1+x}{1-x}\right)\right)dx$ equal to [2002S]
(a) $-\frac{1}{2}$ (b) 0 (c) 1 (d) $2\ell n\left(\frac{1}{2}\right)$

17. Let $T > 0$ be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x + T) = f(x)$.
If $I = \int_0^T f(x)dx$ then the value of $\int_3^{3+3T} f(2x)dx$ is [2002S]
(a) $3/2I$ (b) $2I$ (c) $3I$ (d) $6I$

18. Let $f(x) = \int_1^x \sqrt{2-t^2}\,dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are [2002S]
(a) ± 1 (b) $\pm\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{2}$ (d) 0 and 1

19. Let $f: (0, \infty) \to R$ and $F(x) = \int_0^x f(t)dt$. If $F(x^2) = x^2(1 + x)$, then $f(4)$ equals [2001S]
(a) 5/4 (b) 7 (c) 4 (d) 2

20. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}dx$, $a > 0$, is [2001S]
(a) π (b) $a\pi$ (c) $\pi/2$ (d) 2π

21. The value of the integral $\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right|dx$ is: [2000S]
(a) 3/2 (b) 5/2 (c) 3 (d) 5

22. If $f(x) = \begin{cases} e^{\cos x}\sin x, & \text{for } |x| \le 2 \\ 2, & \text{otherwise,} \end{cases}$ then $\int_{-2}^{3} f(x)dx =$ [2000S]
(a) 0 (b) 1 (c) 2 (d) 3

23. Let $g(x) = \int_0^x f(t)dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$, for $t \in [0,1]$ and $0 \le f(t) \le \frac{1}{2}$, for $t \in [1,2]$.
Then $g(2)$ satisfies the inequality [2000S]
(a) $-\frac{3}{2} \le g(2) < \frac{1}{2}$ (b) $0 \le g(2) < 2$
(c) $\frac{3}{2} < g(2) \le \frac{5}{2}$ (d) $2 < g(2) < 4$

24. If for a real number y, $[y]$ is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\,dx$ is [1999 - 2 Marks]
(a) $-\pi$ (b) 0 (c) $-\pi/2$ (d) $\pi/2$

25. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to [1999 - 2 Marks]
(a) 2 (b) –2 (c) 1/2 (d) –1/2

26. If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ equals [1997 - 2 Marks]
(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$
(c) $g(x)\,g(\pi)$ (d) $\frac{g(x)}{g(\pi)}$

27. The value of $\int_{\pi}^{2\pi} [2\,Sinx]\,dx$ where [.] represents the greatest integer function is [1995S]
(a) $\frac{-5\pi}{3}$ (b) $-\pi$ (c) $\frac{5\pi}{3}$ (d) -2π

28. If $f(x) = A\sin\left(\frac{\pi x}{2}\right) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)dx = \frac{2A}{\pi}$, then constants A and B are [1995S]
(a) $\frac{\pi}{2}$ and $\frac{\pi}{2}$ (b) $\frac{2}{\pi}$ and $\frac{3}{\pi}$
(c) 0 and $\frac{-4}{\pi}$ (d) $\frac{4}{\pi}$ and 0

29. The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is [1993 - 1 Marks]
(a) 0 (b) 1 (c) $\pi/2$ (d) $\pi/4$

30. Let $f: R \to R$ and $g: R \to R$ be continous functions. Then the value of the integral
$\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)]\,[g(x) - g(-x)]dx$ is [1990 - 2 Marks]
(a) π (b) 1 (c) -1 (d) 0

31. For any integer n the integral —
$\int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx$ has the value [1985 - 2 Marks]
(a) π (b) 1 (c) 0 (d) none of these

32. The value of the integral $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$ is [1983 - 1 Mark]
(a) $\pi/4$ (b) $\pi/2$
(c) π (d) none of these

33. Let a, b, c be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx$$

Then the quadratic equation $ax^2+bx+c=0$ has **[1981 - 2 Marks]**

(a) no root in (0, 2)
(b) at least one root in (0, 2)
(c) a double root in (0, 2)
(d) two imaginary roots

34. The value of the definite integral $\int_0^1 (1+e^{-x^2})\,dx$ is **[1981 - 2 Marks]**

(a) -1 (b) 2
(c) $1+e^{-1}$ (d) none of these

2 Integer Value Answer/ Non-Negative Integer

35. The greatest integer less than or equal to

$$\int_1^2 \log_2(x^3+1)\,dx + \int_1^{\log_2 9} (2^x-1)^{\frac{1}{3}}\,dx$$

is ______. **[Adv. 2022]**

36. For any real number x, let $[x]$ denote the largest integer less than or equal to x. If $I = \int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$, then the value of $9I$ is ____. **[Adv. 2021]**

37. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.

If $F:[0,\pi]\to\mathbb{R}$ is defined by $F(x) = \int_0^x f(t)\,dt$, and if

$$\int_0^{\pi} (f'(x)+F(x))\cos x\,dx = 2,$$

then the value of $f(0)$ is ______ **[Adv. 2020]**

38. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$, then $27I^2$ equals ____. **[Adv. 2019]**

39. The value of the integral $\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{\frac{1}{4}}}\,dx$ is ____. **[Adv. 2018]**

40. Let $f: R\to R$ be a differentiable function such that $f(0)=0$, $f\left(\frac{\pi}{2}\right)=3$ and $f'(0)=1$.

If $g(x) = \int_x^{\frac{\pi}{2}} [f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]\,dt$ for $x\in\left(0,\frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$ ______. **[Adv. 2018]**

41. The total number of distinct $x\in[0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}\,dt = 2x-1$ is **[Adv. 2016]**

42. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is **[Adv. 2015]**

43. Let $f: R\to R$ be a function defined by $f(x) = \begin{cases}[x], & x\le 2\\ 0, & x>2\end{cases}$ where $[x]$ is the greatest integer less than or equal to x, if $I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}\,dx$, then the value of $(4I-1)$ is **[Adv. 2015]**

44. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$ is **[Adv. 2014]**

45. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd,}\\ 1+[x]-x & \text{if } [x] \text{ is even}\end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\,dx$ is **[2010]**

46. Let f: **R** → **R** be a continuous function which satisfies

$$f(x) = \int_0^x f(t)\,dt.$$

Then the value of $f(\ln 5)$ is **[2009]**

3 Numeric/ New Stem Based Questions

47. Let $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$, $i = 1, 2$, and $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$ be functions such that $g_1(x) = 1$, $g_2(x) = |4x-\pi|$ and $f(x) = \sin^2 x$, for all $x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$

Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x).g_i(x)\,dx,\ i=1,2$

The value of $\frac{16S_1}{\pi}$ is ____. **[Adv. 2021]**

48. The value of $\frac{48S_2}{\pi^2}$ is ____. **[Adv. 2021]**

49. The value of the integral

$$\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{(\sqrt{\cos\theta}+\sqrt{\sin\theta})^5}\,d\theta$$ equals ______ **[Adv. 2019]**

4 Fill in the Blanks

50. Let $\frac{d}{dx}F(x)=\frac{e^{\sin x}}{x}, x>0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}dx = F(k)-F(1)$ then one of the possible values of k is .... [1997 - 2 Marks]

51. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is ______. [1997 - 2 Marks]

52. For $n>0$, $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx =$ ______. [1996 - 1 Mark]

53. If for nonzero x, $af(x)+bf\left(\frac{1}{x}\right)=\frac{1}{x}-5$ where $a \neq b$, then $\int_1^2 f(x)\,dx =$ ______. [1996 - 2 Marks]

54. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ is ______. [1994 - 2 Marks]

55. The value of $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi}d\phi$ is ______. [1993 - 2 Marks]

56. The value of $\int_{-2}^{2} |1-x^2|\,dx$ is ______. [1989 - 2 Marks]

57. The integral $\int_0^{1.5} [x^2]dx$, where [ ] denotes the greatest integer function, equals ______. [1988 - 2 Marks]

58. $f(x)=\begin{vmatrix} \sec x & \cos x & \sec^2 x+\cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$.

Then $\int_0^{\pi/2} f(x)\,dx =$ ______. [1987 - 2 Marks]

5 True / False

59. The value of the integral $\int_0^{2a} \left[\frac{f(x)}{\{f(x)+f(2a-x)\}}\right]dx$ is equal to a. [1988 - 1 Mark]

6 MCQs with One or More than One Correct Answer

60. Consider the equation [Adv. 2022]

$$\int_1^e \frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2}dx = 1, \quad a \in (-\infty, 0)\cup(1,\infty).$$

Which of the following statements is/are TRUE?
(a) No a satisfies the above equation
(b) An integer a satisfies the above equation
(c) An irrational number a satisfies the above equation
(d) More than one a satisfy the above equation

61. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that

$f(0)=1$ and $\int_0^{\frac{\pi}{3}} f(t)dt = 0$

Then which of the following statements is (are) TRUE? [Adv. 2021]
(a) The equation $f(x)-3\cos 3x=0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$
(b) The equation $f(x)-3\sin 3x=-\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$
(c) $\lim_{x\to\infty}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$
(d) $\lim_{x\to\infty}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

62. Which of the following inequalities is/are TRUE? [Adv. 2020]
(a) $\int_0^1 x\cos x\,dx \ge \frac{3}{8}$ (b) $\int_0^1 x\sin x\,dx \ge \frac{3}{10}$
(c) $\int_0^1 x^2\cos x\,dx \ge \frac{1}{2}$ (d) $\int_0^1 x^2\sin x\,dx \ge \frac{2}{9}$

63. Let $f: R\to R$ be given by $f(x)=(x-1)(x-2)(x-5)$. Define

$F(x)=\int_0^x f(t)dt, x>0$.

Then which of the following options is/are correct? [Adv. 2019]
(a) F has a local maximum at $x=2$
(b) F has a local minimum at $x=1$
(c) F has two local maxima and one local minimum in $(0,\infty)$
(d) $F(x)\ne 0$ for all $x\in(0,5)$

64. If $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$, then [Adv. 2017]
(a) $I>\log_e 99$ (b) $I<\log_e 99$
(c) $I<\frac{49}{50}$ (d) $I>\frac{49}{50}$

65. If $g(x)=\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)dt$, then [Adv. 2017]
(a) $g'\left(\frac{\pi}{2}\right)=-2\pi$ (b) $g'\left(-\frac{\pi}{2}\right)=2\pi$
(c) $g'\left(\frac{\pi}{2}\right)=2\pi$ (d) $g'\left(-\frac{\pi}{2}\right)=-2\pi$

66. Let f: R→(0,1) be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)? [Adv. 2017]

(a) $x^9 - f(x)$ (b) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

(c) $e^x - \int_0^x f(t)\sin t\,dt$ (d) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$

67. Let $f'(x) = \dfrac{192x^3}{2+\sin^4 \pi x}$ for all $x \in \mathbb{R}$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are [Adv. 2015]

(a) $m=13, M=24$ (b) $m=\frac{1}{4}, M=\frac{1}{2}$

(c) $m=-11, M=0$ (d) $m=1, M=12$

68. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are) [Adv. 2015]

(a) $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ (b) $\int_0^{\pi/4} f(x)dx = 0$

(c) $\int_0^{\pi/4} xf(x)dx = \frac{1}{6}$ (d) $\int_0^{\pi/4} f(x)dx = 1$

69. The option(s) with the values of a and L that satisfy the following equation is(are) [Adv. 2015]

$$\frac{\int_0^{4\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt}{\int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt} = L?$$

(a) $a=2, L=\dfrac{e^{4\pi}-1}{e^{\pi}-1}$ (b) $a=2, L=\dfrac{e^{4\pi}+1}{e^{\pi}+1}$

(c) $a=4, L=\dfrac{e^{4\pi}-1}{e^{\pi}-1}$ (d) $a=4, L=\dfrac{e^{4\pi}+1}{e^{\pi}+1}$

70. Let $f: (0,\infty) \to R$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$. Then [Adv. 2014]

(a) $f(x)$ is monotonically increasing on $[1,\infty)$

(b) $f(x)$ is monotonically decreasing on (0, 1)

(c) $f(x)+f\left(\frac{1}{x}\right) = 0$, for all $x \in (0,\infty)$

(d) $f(2^x)$ is an odd function of x on R

71. Let $f: [a, b] \to [1,\infty)$ be a continuous function and let $g: R \to R$ be defined as [Adv. 2014]

$$g(x) = \begin{cases} 0, & \text{if } x < a, \\ \int_a^x f(t)dt, & \text{if } a \le x \le b, \\ \int_a^b f(t)dt, & \text{if } x > b. \end{cases}$$ then

(a) $g(x)$ is continuous but not differentiable at a

(b) $g(x)$ is differentiable on R

(c) $g(x)$ is continuous but not differentiable at b

(d) $g(x)$ is continuous and differentiable at either (a) or (b) but not both

72. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true? [2010]

(a) $f''(x)$ exists for all $x \in (0,\infty)$

(b) $f'(x)$ exists for all $x \in (0,\infty)$ and f' is continuous on $(0,\infty)$, but not differentiable on $(0,\infty)$

(c) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha,\infty)$

(d) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0,\infty)$

73. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are) [2010]

(a) $\frac{22}{7} - \pi$ (b) $\frac{2}{105}$

(c) 0 (d) $\frac{71}{15} - \frac{3\pi}{2}$

74. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$ $n = 0, 1, 2, \ldots$, then [2009]

(a) $I_n = I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(c) $\sum_{m=1}^{10} I_{2m} = 0$ (d) $I_n = I_{n+1}$

75. Let $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2-e^{x-1}, & 1 < x \le 2 \\ x-e, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt, x \in [0, 3]$ then $g(x)$ has [2006 - 5M, –1]

(a) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$

(b) local maxima at $x = 1$ and local minima at $x = 2$

(c) no local maxima

(d) no local minima

76. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3 (t-3)^5 \, dt$ has a local minimum at $x =$ [1999 - 3 Marks]

(a) 0 (b) 1 (c) 2 (d) 3

77. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then, [2008]

(a) $f''(x)$ vanishes at least twice on [0, 1]

(b) $f'\left(\frac{1}{2}\right) = 0$

(c) $\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right) \sin x \, dx = 0$

(d) $\int_{0}^{1/2} f(t) e^{\sin \pi t} dt = \int_{1/2}^{1} f(1-t) e^{\sin \pi t} dt$

78. Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x. Then $\int_{-1}^{1} f(x)\,dx$ is [1998 - 2 Marks]

(a) 1 (b) 2

(c) 0 (d) 1/2

79. If $\int_0^x f(t)\,dt = x + \int_x^1 t f(t)\,dt$, then the value of $f(1)$ is [1998 - 2 Marks]

(a) 1/2 (b) 0 (c) 1 (d) –1/2

7 Match the Following

80. **List - I**

P. The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)\,dx = 1$, is

Q. The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is

R. $\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals

S. $\dfrac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_{0}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$

List - II

1. 8
2. 2
3. 4
4. 0

[Adv. 2014]

	P	Q	R	S
(a)	3	2	4	1
(b)	2	3	4	1
(c)	3	2	1	4
(d)	2	3	1	4

8 Comprehension/Passage Based Questions

PASSAGE - 1

Let $f : \left[0, \frac{\pi}{2}\right] \to [0,1]$ be the function defined by $f(x) = \sin^2 x$ and let $g : \left[0, \frac{\pi}{2}\right] \to [0, \infty)$ be the function defined by $g(x) = \sqrt{\frac{\pi x}{2} - x^2}$.

81. The value of $2\int_0^{\frac{\pi}{2}} f(x)g(x)\,dx - \int_0^{\frac{\pi}{2}} g(x)\,dx$ is ______.

[Adv. 2024]

82. The value of $\frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} f(x)g(x)\,dx$ is ______.

[Adv. 2024]

PASSAGE - 2

Let $\psi_1 : [0,\infty) \to \mathbb{R}$, $\psi_2 : [0,\infty) \to \mathbb{R}$, $f : [0,\infty) \to \mathbb{R}$, and $g : [0,\infty) \to \mathbb{R}$, be functions such that $f(0) = f(0) = 0$,

$\psi_1(x) = e^{-x} + x, \; x \ge 0,$

$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \; x \ge 0,$

$f(x) = \int_{-x}^{x} (|t| - t^2) e^{-t^2} dt, \; x > 0$

and $g(x) = \int_0^{x^2} \sqrt{t} e^{-t} dt, \; x > 0$ [Adv. 2021]

83. Which of the following statements is TRUE ?
(a) $f(\sqrt{\ln 3})+g(\sqrt{\ln 3})=\frac{1}{3}$
(b) For every $x>1$, there exists an $\alpha \in (1, x)$ such that $\psi_1(x)=1+\alpha x$
(c) For every $x>0$, there exists a $\beta \in (0, x)$ such that $\psi_2(x)=2x(\psi_1(\beta)-1)$
(d) f is an increasing function on the interval $\left[0,\frac{3}{2}\right]$

84. Which of the following statements is TRUE ?
(a) $\psi_1(x)\le 1$, for all $x>0$
(b) $\psi_2(x)\le 0$, for all $x>0$
(c) $f(x)\ge 1-e^{-x^2}-\frac{2}{3}x^3+\frac{2}{5}x^5$, for all $x\in\left(0,\frac{1}{2}\right)$
(d) $g(x)\le \frac{2}{3}x^3-\frac{2}{5}x^5+\frac{1}{7}x^7$, for all $x\in\left(0,\frac{1}{2}\right)$

PASSAGE - 3

Let $F:\mathbb{R}\to\mathbb{R}$ be a thrice differentiable function. Suppose that $F(1)=0, F(3)=-4$ and $F'(x)<0$ for all $x\in\left(\frac{1}{2},3\right)$. Let $f(x)=xF(x)$ for all $x\in\mathbb{R}$. **[Adv. 2015]**

85. The correct statement(s) is(are)
(a) $f'(1)<0$
(b) $f(2)<0$
(c) $f'(x)\neq 0$ for any $x\in(1,3)$
(d) $f'(x)=0$ for some $x\in(1,3)$

86. If $\int_1^3 x^2F'(x)dx=-12$ and $\int_1^3 x^3F''(x)dx=40$, then the correct expression(s) is (are)
(a) $9f'(3)+f'(1)-32=0$ (b) $\int_1^3 f(x)dx=12$
(c) $9f'(3)-f'(1)+32=0$ (d) $\int_1^3 f(x)dx=-12$

PASSAGE - 4

Let $f(x)=(1-x)^2\sin^2 x+x^2$ for all $x\in IR$ and let $g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)dt$ for all $x\in(1,\infty)$. **[2012]**

87. Consider the statements:
P: There exists some $x\in R$ such that, $f(x)+2x=2(1+x^2)$
Q: There exists some $x\in R$ such that, $2f(x)+1=2x(1+x)$
Then
(a) both P and Q are true
(b) P is true and Q is false
(c) P is false and Q is true
(d) both P and Q are false

88. Which of the following is true?
(a) g is increasing on $(1,\infty)$
(b) g is decreasing on $(1,\infty)$
(c) g is increasing on $(1, 2)$ and decreasing on $(2,\infty)$
(d) g is decreasing on $(1, 2)$ and increasing on $(2,\infty)$

PASSAGE - 5

Consider the function $f:(-\infty,\infty)\to(-\infty,\infty)$ defined by $f(x)=\frac{x^2-ax+1}{x^2+ax+1}, 0<a<2.$ **[2008]**

89. Which of the following is true?
(a) $(2+a)^2f''(1)+(2-a)^2f''(-1)=0$
(b) $(2-a)^2f''(1)-(2+a)^2f''(-1)=0$
(c) $f'(1)f'(-1)=(2-a)^2$
(d) $f'(1)f'(-1)=-(2+a)^2$

90. Which of the following is true?
(a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x=1$
(b) $f(x)$ is increasing on $(-1, 1)$ and has a local minimum at $x=1$
(c) $f(x)$ is increasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x=1$
(d) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x=1$

91. Let $g(x)=\int_0^{e^x}\frac{f'(t)}{1+t^2}dt$. Which of the following is true? **[2008]**
(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0,\infty)$
(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0,\infty)$
(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0,\infty)$
(d) $g'(x)$ does not change sign on $(-\infty,\infty)$

PASSAGE - 6

Let the definite integral be defined by the formula $\int_a^b f(x)dx=\frac{b-a}{2}(f(a)+f(b))$. For more accurate result for $c\in(a,b)$, we can use $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx=F(c)$ so that for $c=\frac{a+b}{2}$, we get $\int_a^b f(x)dx=\frac{b-a}{4}(f(a)+f(b)+2f(c))$.

92. $\int_0^{\pi/2}\sin x\,dx=$ **[2006 - 5M, –2]**
(a) $\frac{\pi}{8}\left(1+\sqrt{2}\right)$ (b) $\frac{\pi}{4}\left(1+\sqrt{2}\right)$
(c) $\frac{\pi}{8\sqrt{2}}$ (d) $\frac{\pi}{4\sqrt{2}}$

93. If $\lim_{x \to a} \dfrac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$, then $f(x)$ is of maximum degree

(a) 4 (b) 3 (c) 2 (d) 1

94. If $f''(x) < 0 \ \forall x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to

(a) $\dfrac{f(b)-f(a)}{b-a}$ (b) $\dfrac{2(f(b)-f(a))}{b-a}$

(c) $\dfrac{2f(b)-f(a)}{2b-a}$ (d) 0

10 Subjective Problems

95. Find the value of $\displaystyle\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)} dx$ **[2004 - 4 Marks]**

96. If $y(x) = \displaystyle\int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$, then find $\dfrac{dy}{dx}$ at $x = \pi$ **[2004 - 2 Marks]**

97. If f is an even function then prove that

$$\int_0^{\pi/2} f(\cos 2x) \cos x\, dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x) \cos x\, dx.$$

[2003 - 2 Marks]

98. Let $f(x), x \geq 0$, be a non–negative continuous function, and let $F(x) = \int_0^x f(t)\,dt, x \geq 0$. If for some $c > 0$, $f(x) \leq cF(x)$ for all $x \geq 0$, then show that $f(x) = 0$ for all $x \geq 0$. **[2001 - 5 Marks]**

99. For $x>0$, let $f(x) = \displaystyle\int_1^x \frac{\ln t}{1+t} dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$. Here, $\ln t = \log_e t$. **[2000 - 5 Marks]**

100. Integrate $\displaystyle\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx.$ **[1999 - 5 Marks]**

101. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2\int_0^1 \tan^{-1} x\, dx.$

Hence or otherwise, evaluate the integral

$\int_0^1 \tan^{-1}(1-x+x^2)\, dx.$ **[1998 - 8 Marks]**

102. Let $a + b = 4$, where $a < 2$, and let $g(x)$ be a differentiable function. If $\dfrac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x)\,dx + \int_0^b g(x)dx$ increases as $(b - a)$ increases. **[1997 - 5 Marks]**

103. Determine the value of $\displaystyle\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx.$ **[1997 - 5 Marks]**

104. Evaluate the definite integral :

$\displaystyle\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right) \cos^{-1}\left(\frac{2x}{1+x^2}\right) dx$ **[1995 - 5 Marks]**

105. Evalute $\displaystyle\int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} dx.$ **[1993 - 5 Marks]**

106. Determine a positive integer $n \leq 5$, such that

$\displaystyle\int_0^1 e^x (x-1)^n\, dx = 16 - 6e$ **[1992 - 4 Marks]**

107. Evaluate $\displaystyle\int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi} dx$ **[1991 - 4 Marks]**

108. If 'f' is a continuous function with $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2$! **[1991 - 4 Marks]**

109. Prove that for any positive integer k,

$$\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + \ldots\ldots\ldots\ldots + \cos(2k-1)x]$$

Hence prove that $\displaystyle\int_0^{\pi/2} \sin 2kx \cot x\, dx = \frac{\pi}{2}$ **[1990 - 4 Marks]**

110. Show that $\displaystyle\int_0^{\pi/2} f(\sin 2x) \sin x\, dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x) \cos x\, dx$ **[1990 - 4 Marks]**

111. If f and g are continuous function on $[0, a]$ satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, then show that $\displaystyle\int_0^a f(x)g(x)dx = \int_0^a f(x)dx$ **[1989 - 4 Marks]**

112. Evaluate $\displaystyle\int_0^1 \log[\sqrt{1-x} + \sqrt{1+x}]dx$ **[1988 - 5 Marks]**

113. Investigate for maxima and minima the function **[1988 - 5 Marks]**

$$f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]dt$$

114. Evaluate : $\int_0^{\pi} \frac{x\,dx}{1+\cos\alpha\sin x}, 0<\alpha<\pi$ [1986 - 2½ Marks]

115. Evaluate the following : $\int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ [1985 - 2½ Marks]

116. Given a function $f(x)$ such that [1984 - 4 Marks]
(i) it is integrable over every interval on the real line and
(ii) $f(t+x)=f(x)$, for every x and a real t, then show that the integral $\int_a^{a+t} f(x)\,dx$ is independent of a.

117. Evaluate the following $\int_0^{\frac{1}{2}} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ [1984 - 2 Marks]

118. Evaluate : $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}dx$ [1983 - 3 Marks]

119. Find the value of $\int_{-1}^{3/2} |x\sin\pi x|\,dx$ [1982 - 3 Marks]

120. Show that $\int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$. [1982 - 2 Marks]

Topic-4: Summation of Series by Integration

1 MCQs with One Correct Answer

1. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt$ is [2010]

(a) 0 (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{64}$

2. If $I(m,n)=\int_0^1 t^m(1+t)^n\,dt$, then the expression for I(m,n) in terms of $I(m+1,n-1)$ is [2003S]

(a) $\frac{2^n}{m+1}-\frac{n}{m+1}I(m+1,n-1)$

(b) $\frac{n}{m+1}I(m+1,n-1)$

(c) $\frac{2^n}{m+1}+\frac{n}{m+1}I(m+1,n-1)$

(d) $\frac{m}{n+1}I(m+1,n-1)$

2 Integer Value Answer/Non-Negative Integer

3. Let the function $f: [1,\infty)\to\mathbb{R}$ be defined by

$$f(t)=\begin{cases}(-1)^{n+1}2, & \text{if } t=2n-1, n\in\mathbb{N},\\ \frac{(2n+1-t)}{2}f(2n-1)+ & \text{if } 2n-1<t<2n+1,\\ \frac{(t-(2n-1))}{2}f(2n+1), & n\in\mathbb{N}.\end{cases}$$

Define $g(x)=\int_1^x f(t)dt, x\in(1,\infty)$. Let α denote the number of solutions of the equation $g(x)=0$ in the interval (1, 8] and $\beta=\lim_{x\to 1^+}\frac{g(x)}{x-1}$. Then the value of $\alpha+\beta$ is equal to ______. [Adv. 2024]

4. For $x\in\mathbb{R}$, let $\tan^{-1}(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the minimum value of the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\int_0^{x\tan^{-1}x}\frac{e^{(t-\cos t)}}{1+t^{2023}}dt$ is [Adv. 2023]

3 Numeric/ New Stem Based Questions

Question Stem

Let $f_1:(0,\infty)\to\mathbb{R}$ and $f_2:(0,\infty)\to\mathbb{R}$ be defined by

$$f_1(x)=\int_0^x\prod_{j=1}^{21}(t-j)^j\,dt,\ x>0$$

and $f_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450, x>0$,
where for any positive integer n and real numbers $a_1, a_2, ..., a_n$, $\Pi_{i=1}^n a_i$ denotes the product of $a_1, a_2, ..., a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0,\infty)$

5. The value of $2m_1+3n_1+m_1n_1$ is ______. [Adv. 2021]
6. The value of $6m_2+4n_2+8m_2n_2$ is ______. [Adv. 2021]
7. For each positive integer n, let

$$y_n=\frac{1}{n}(n+1)(n+2)...(n+n)^{\frac{1}{n}}$$

For $x\in\mathbb{R}$, let $[x]$ be the greatest integer less than or equal to x. If $\lim_{n\to\infty}y_n=L$, then the value of $[L]$ is ______. [Adv. 2018]

8. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t|f(f(t))|\,dt$ for all $x \in [-1, 2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **[Adv. 2015]**

9. For, $a \in R, |a| > 1$, let

$$\lim_{x\to\infty}\left(\frac{1+\sqrt[3]{2}+.....+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+....+\frac{1}{(an+n)^2}\right)}\right) = 54$$

Then the possible value(s) of a is/are **[Adv. 2019]**

(a) -9 (b) 7 (c) -6 (d) 8

6 MCQs with One or More than One Correct Answer

10. Let $f(x) = \lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)...\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)...\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all $x > 0$. Then **[Adv. 2016]**

(a) $f\left(\frac{1}{2}\right) \geq f(1)$ (b) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$

(c) $f'(2) \leq 0$ (d) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

8 Comprehension/Passage Based Questions

Given that for each $a \in (0, 1)$, $\lim_{h\to 0^+}\int_{h}^{1-h} t^{-a}(1-t)^{a-1}dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$. **[Adv. 2014]**

11. The value of $g\left(\frac{1}{2}\right)$ is

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

12. The value of $g'\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{2}$ (b) π (c) $-\frac{\pi}{2}$ (d) 0

10 Subjective Problems

13. Let $I_m = \int_0^{\pi}\frac{1-\cos mx}{1-\cos x}dx$. Use mathematical induction to prove that $I_m = m\pi, m = 0, 1, 2, ...$ **[1995 - 5 Marks]**

14. Show that $\int_0^{n\pi+v}|\sin x|\,dx = 2n+1-\cos v$ where n is a positive integer and $0 \leq v < \pi$. **[1994 - 4 Marks]**

15. Show that : $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{6n}\right) = \log 6$ **[1981 - 2 Marks]**

Answer Key

Topic-1 : Standard Integrals, Integration by Substitution, Integration by Parts

1. (c) 2. (c) 3. (c) 4. $\frac{-3}{2}, \frac{35}{36}, R$ 5. (a, c) 6. (a, b, d) 7. (d)

Topic-2 : Integration of the Forms: $\int e^x(f(x) + f'(x))dx$, $\int e^{kx}(df(x) + f'(x))dx$, Integration by Partial Fractions, Integration of Different Expressions of e^x

1. (d)

Topic-3 : Evaluation of Definite Integral by Substitution, Properties of Definite Integrals

1. (c)	2. (b)	3. (a)	4. (a)	5. (b)	6. (d)	7. (b)	8. (a)	9. (b)	10. (c)
11. (c)	12. (c)	13. (b)	14. (a)	15. (d)	16. (a)	17. (c)	18. (a)	19. (c)	20. (c)
21. (b)	22. (c)	23. (b)	24. (c)	25. (a)	26. (a)	27. (a)	28. (d)	29. (d)	30. (d)
31. (c)	32. (a)	33. (b)	34. (d)	35. (5)	36. (182)	37. (4)	38. (4)	39. (2)	40. (2)
41. (1)	42. (9)	43. (0)	44. (2)	45. (4)	46. (0)	47. (2)	48. (1.5)	49. (0.5)	50. (16)

51. (2) 52. (π^2) 53. $\frac{1}{a^2-b^2}\left[a(\log 2-5)+\frac{7b}{2}\right]$ 54. $\frac{1}{2}$ 55. $\pi(\sqrt{2}-1)$ 56. (4) 57. $2-\sqrt{2}$

58. $-\left(\frac{15\pi+32}{60}\right)$ 59. (True) 60. (c, d) 61. (a,b,c) 62. (a,b,d) 63. (a, b, d) 64. (b, d) 65. (Bonus)

66. (a, b) 67. (d) 68. (a, b) 69. (a, c) 70. (a, c, d) 71. (a, c) 72. (b, c) 73. (a) 74. (a, b, c)
75. (a, b) 76. (b, d) 77. (a, b, c, d) 78. (a) 79. (a) 80. (d) 81. (0) 82. (0.25) 83. (c) 84. (d)
85. (a, b, c) 86. (c, d) 87. (c) 88. (b) 89. (a) 90. (a) 91. (b) 92. (a) 93. (d) 94. (b)

Topic-4 : Reduction Formulae for Definite Integration, Gamma & Beta Function, Walli's Formula, Summation of Series by Integration

1. (b) 2. (a) 3. (5) 4. (0) 5. (57) 6. (6) 7. (1) 8. (7) 9. (a, d) 10. (b, c)
11. (a) 12. (d)

Hints & Solutions

Topic-1: Standard Integrals, Integration by Substitution, Integration by Parts

1. **(c)** $I=\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$

Let $\sec x+\tan x=t \Rightarrow \sec x-\tan x=\frac{1}{t}$

$\Rightarrow \sec x=\frac{1}{2}\left(t+\frac{1}{t}\right)$ and $\sec x(\sec x+\tan x)\,dx=dt$

$\Rightarrow \sec x\,dx=\frac{dt}{t}$

$\therefore \quad I=\frac{1}{2}\int\frac{\left(t+\frac{1}{t}\right)dt}{t^{9/2}.t}=\frac{1}{2}\int\left(t^{-9/2}+t^{-13/2}\right)dt$

$=\frac{-1}{7}t^{-7/2}-\frac{1}{11}t^{-11/2}+K$

$=-\frac{1}{7t^{7/2}}-\frac{1}{11t^{11/2}}+K=-\frac{1}{t^{11/2}}\left(\frac{1}{11}+\frac{t^2}{7}\right)+K$

$=\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

2. **(c)** Given $I=\int\frac{e^x}{e^{4x}+e^{2x}+1}dx,$

$J=\int\frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx=\int\frac{e^{3x}}{e^{4x}+e^{2x}+1}dx$

$\therefore \quad J-I=\int\frac{e^x(e^{2x}-1)}{e^{4x}+e^{2x}+1}dx$

Let $e^x=t \Rightarrow e^x dx=dt$

$\therefore \quad J-I=\int\frac{t^2-1}{t^4+t^2+1}dt=\int\frac{1-\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}dt$

Let $t+\frac{1}{t}=u \Rightarrow \left(1-\frac{1}{t^2}\right)dt=du$

$\therefore \quad J-I=\int\frac{du}{u^2-1}=\frac{1}{2}\log\left|\frac{u-1}{u+1}\right|+c$

$=\frac{1}{2}\log\left|\frac{\frac{t^2+1}{t}-1}{\frac{t^2+1}{t}+1}\right|+c=\frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right|+c$

3. **(c)** $I=\int\frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx=\int\frac{(\cos^2 x+\cos^4 x)\cos x}{\sin^2 x(1+\sin^2 x)}dx$

$=\int\frac{[1-\sin^2 x+(1-\sin^2 x)^2]\cos x}{\sin^2 x(1+\sin^2 x)}dx$

$=\int\frac{(2-3\sin^2 x+\sin^4 x)\cos x}{\sin^2 x(1+\sin^2 x)}dx$

Put $\sin x=t \Rightarrow \cos x\,dx=dt$

$I=\int\frac{2-3t^2+t^4}{t^4+t^2}dt,\ I=\int\left(1+\frac{2}{t^2}-\frac{6}{t^2+1}\right)dt$

$=t-\frac{2}{t}-6\tan^{-1}t+c$

$=\sin x-2(\sin x)^{-1}-6\tan^{-1}(\sin x)+c$

4. $\int\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx=Ax+B\,ln(9e^{2x}-4)+C$

$\Rightarrow \frac{d}{dx}[Ax+B\ln(9e^{2x}-4)+C]=\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}$

$=A+B.\frac{18e^{2x}}{9e^{2x}-4}$

$\Rightarrow A+\frac{18Be^x}{9e^x-4e^{-x}}=\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}$

$\Rightarrow \frac{(9A+18B)e^x-4Ae^{-x}}{9e^x-4e^{-x}}=\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}$

$\Rightarrow 9A+18B=4;\ -4A=6 \Rightarrow A=\frac{-3}{2};$

$B=\left(4+\frac{27}{2}\right)\frac{1}{18}=\frac{35}{36}$ and C can have any real value.

5. **(a, c)** Given that $f'(x)=\frac{f(x)}{b^2+x^2}$

$\int\frac{f'(x)}{f(x)}dx=\int\frac{dx}{x^2+b^2}$

$\Rightarrow ln|f(x)|=\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+c$

Now $f(0)=1$

$\therefore c=0$

$\therefore |f(x)|=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)} \Rightarrow f(x)=\pm e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\because f(0)=1 \therefore f(x)=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\because f'(x)=e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}\times\frac{1}{b^2+x^2}>0$ for all value of b.

$\therefore f(x)$ is increasing function for all values of b.
So, option (a) is true.

$f(-x)=e^{-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\therefore f(x).f(-x)=e^0=1$
So, option (c) is true.

6. (a, b, d) $\lim\limits_{x\to0}f(x)=\lim\limits_{x\to0}x\cdot\lim\limits_{x\to0}g(x)=0$

$$\because\ f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow\ f'(x)=\lim_{h\to0}\frac{f(x)+f(h)+f(x)\cdot f(h)-f(x)}{h}$$

$$\Rightarrow\ f'(x)=\lim_{h\to0}\frac{f(h)}{h}(1+f(x))$$

$$\left[\because\lim_{h\to0}\frac{f(h)}{h}=\lim_{h\to0}\frac{hg(h)}{h}=0\right]$$

$\Rightarrow\ f'(x)=1+f(x)$

$$\Rightarrow\ \frac{f'(x)}{1+f(x)}=1\Rightarrow\int\frac{f'(x)}{1+f(x)}dx=\int dx$$

$\Rightarrow\ \ln(1+f(x))=x+c$
$\Rightarrow\ 1+f(x)=e^{x+c}\ (\because f(0)=0\Rightarrow c=0)$
$\Rightarrow\ f(x)=e^x-1\ \Rightarrow\ f'(x)=e^x$
So, $f(x)$ is differentiable at every $x\in R$
So, option (a) is true.
$f'(0)=e^0=1$
So, option (d) is true.

$$g(x)=\frac{f(x)}{x}=\begin{cases}\dfrac{e^x-1}{x} & ;\ x\neq0\\ 1 & ;\ x=0\end{cases}$$

$\therefore$ $g(x)$ is differentiable at every $x\in R$
So, option (b) is true.

7. **(d)** $F(x)=\int\sin^2x\,dx=\frac{1}{2}\int(1-\cos2x)dx$

$$=\frac{1}{4}(2x-\sin2x)+C$$

Now, $F(x+\pi)=\frac{1}{4}(2x+2\pi-\sin(2x+2\pi))+C$

$$=\frac{1}{4}[2x+2\pi-\sin2x]+C\neq F(x)$$

$\therefore$ Statement -1 is false.
Also $\sin^2(x+\pi)=\sin^2x,\forall x\in R$
$\therefore$ Statement -2 is true.

8. $I=\int(x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}dx$

$$=\int(x^{3m}+x^{2m}+x^m)\left[\frac{2x^{3m}+3x^{2m}+6x^m}{x^m}\right]^{1/m}dx$$

$$=\int\left(\frac{x^{3m}+x^{2m}+x^m}{x}\right)(2x^{3m}+3x^{2m}+6x^m)^{1/m}dx$$

Put $2x^{3m}+3x^{2m}+6x^m=y$

$\Rightarrow\ 6m(x^{3m-1}+x^{2m-1}+x^{m-1})dx=dy$

$\Rightarrow\ (x^{3m-1}+x^{2m-1}+x^{m-1})dx=\frac{1}{6m}dy$

$$\therefore\ I=\frac{1}{6m}\int y^{1/m}dy=\frac{1}{6m}\left(\frac{y^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right)+c=\frac{1}{6}\left(\frac{y^{\frac{m+1}{m}}}{m+1}\right)+c$$

$$=\frac{1}{6}\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{m+1}+c$$

9. $I=\int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$$=\int\sin^{-1}\left[\frac{x+1}{\sqrt{x^2+2x+\frac{13}{4}}}\right]dx$$

$$=\int\sin^{-1}\left[\frac{x+1}{\sqrt{(x+1)^2)+(3/2)^2}}\right]dx$$

Put $x+1=3/2\tan\theta$, $dx=\frac{3}{2}\sec^2\theta d\theta$

$$\therefore\ I=\int\sin^{-1}\left[\frac{\left(\frac{3}{2}\tan\theta\right)}{\sqrt{\frac{9}{4}\tan^2\theta+\frac{9}{4}}}\right].\frac{3}{2}\sec^2\theta\,d\theta$$

$$=\frac{3}{2}\int\sin^{-1}(\sin\theta).\sec^2\theta d\theta$$

$$=\frac{3}{2}\int\theta\sec^2\theta\,d\theta=\frac{3}{2}\left[\theta\tan\theta-\int\tan\theta\,d\theta\right]$$

$$=\frac{3}{2}[\theta\tan\theta-\log|\sec\theta|]+c$$

$$=\frac{3}{2}\left[\frac{2}{3}(x+1)\tan^{-1}\left[\frac{2}{3}(x+1)\right]-\log\sqrt{1+\frac{4}{9}(x+1)^2]}\right]+c$$

$$=(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right)-\frac{3}{4}\log(9+4x^2+8x+4)+\frac{3}{4}\log9+c$$

$$=(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right)-\frac{3}{4}\log(4x^2+8x+13)+c$$

10. $I=\int\frac{(x+1)}{x(1+xe^x)^2}dx=\int\frac{e^x(x+1)}{xe^x(1+xe^x)^2}dx$

Put $1+xe^x=t\Rightarrow(xe^x+e^x)\,dx=dt$

$\therefore\quad I=\int\frac{dt}{(t-1)t^2}=\int\left(\frac{1}{1-t}+\frac{1}{t}+\frac{1}{t^2}\right)dt$

$=-\log|1-t|+\log|t|-\frac{1}{t}+c$

$=\log\left|\frac{t}{1-t}\right|-\frac{1}{t}+c=\log\left|\frac{1+xe^x}{-xe^x}\right|-\frac{1}{1+xe^x}+c$

$=\log\left(\frac{1+xe^x}{xe^x}\right)-\frac{1}{1+xe^x}+c$

11. Let $I=\int\cos 2\theta\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$

Now we observe that

$\frac{d}{d\theta}\left\{\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right\}$

$=\frac{d}{d\theta}[\ln(\cos\theta+\sin\theta)-\ln(\cos\theta-\sin\theta)]$

$=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}+\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

$=\frac{\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{2}{\cos 2\theta}$

On integrating I with respect to θ, by parts we get

$I=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\int\frac{\sin 2\theta}{2}.\frac{2}{\cos 2\theta}d\theta$

$=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\int\tan 2\theta\,d\theta$

$=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\frac{1}{2}\ln\sec 2\theta+c$

12. $I=\int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}+\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$

$=\int\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx+\int\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}dx$

$I=I_1+I_2$...(i)

where $I_1=\int\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx$ and $I_2=\int\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}dx$

Let $x=y^{12}$ so that $dx=12\,y^{11}\,dy$

$\therefore\quad I_1=\int\frac{12y^{11}}{y^4+y^3}dy=12\int\frac{y^8}{1+y}dy$

$=12\int\left(y^7-y^6+y^5-y^4+y^3-y^2+y-1+\frac{1}{y+1}\right)dy$

$=12\left[\frac{y^8}{8}-\frac{y^7}{7}+\frac{y^6}{6}-\frac{y^5}{5}+\frac{y^4}{4}-\frac{y^3}{3}+\frac{y^2}{2}-y+\log|y+1|\right]+c_1$

$=\frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}+2x^{1/2}-\frac{12}{5}x^{5/12}+3x^{1/3}-4x^{1/4}+6x^{1/6}-12x^{1/12}+12\log|x^{1/12}+1|+c_1$ ...(ii)

Now, $I_2=\int\frac{\ln(1+(x)^{1/6})}{(x)^{1/3}+(x)^{1/2}}dx$

Let $x=z^6$ so that $dx=6z^5\,dz$

$\therefore\quad I_2=\int\frac{\ln(1+z)}{z^2+z^3}.6z^5\,dz=\int\frac{6z^3\ln(z+1)}{z+1}dz$

Put $z+1=t$

$\Rightarrow dz=dt$

$\therefore\quad I_2=\int\frac{6(t-1)^3\ln t}{t}\,dt=6\int(t^2-3t+3-\frac{1}{t})\ln t\,dt$

$=6\left[\int(t^2-3t+3)\ln t\,dt-\int\frac{1}{t}\ln t\,dt\right]$

$=6\left[\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t\right)\ln t-\int\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t\right).\frac{1}{t}dt-\frac{(\ln t)^2}{2}\right]$

$=6\left[\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t\right)\ln t-\int\left(\frac{t^2}{3}-\frac{3}{2}t+3\right)dt-\frac{(\ln t)^2}{2}\right]$

$=6\left[\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t\right)\ln t-\left(\frac{t^3}{9}-\frac{3t^2}{4}+3t\right)-\frac{(\ln t)^2}{2}\right]+c_2$

Now, $t=1+z=1+x^{1/6}$

$\therefore\ I_2=6\left[\left\{\frac{(1+x^{1/6})^3}{3}-\frac{3}{2}(1+x^{1/6})^2+3(1+x^{1/6})\right\}\ln(1+x^{1/6})-\left\{\frac{(1+x^{1/6})^3}{9}-\frac{3}{4}(1+x^{1/6})^2+3(1+x^{1/6})\right\}-\frac{[\ln(1+x^{1/6})^2]}{2}\right]+c_2$ ...(iii)

From (i), (ii) and (iii), we get $I=I_1+I_2$

$\Rightarrow I=\frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/2}+2x^{1/2}-\frac{12}{5}x^{5/12}+3x^{1/13}-4x^{1/4}+6x^{1/6}-12x^{1/12}+12\log\left|x^{1/12}+1\right|$

$+6\left[\left\{\frac{(1+x^{1/6})^3}{3}-\frac{3}{2}(1+x^{1/6})^2+3(1+x^{1/6})\right\}\ln(1+x^{1/6})\right.$

$$-\left\{\frac{\left(1+x^{1/6}\right)^3}{9}-\frac{3}{4}\left(1+x^{1/6}\right)^2+3\left(1+x^{1/6}\right)\right\}$$

$$\left.-\frac{\left[\ln\left(1+x^{1/6}\right)^2\right]}{2}+c\right]$$

13. $I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}dx$

$$=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx\ =\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$

also $(\sin x - \cos x)^2 = t^2 \Rightarrow 1 - \sin 2x = t^2$

$$\Rightarrow \sin 2x = 1-t^2 \quad \therefore I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\sqrt{2}\sin^{-1}t+c\ =\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$$

14. $I=\int\frac{\sqrt{\cos 2x}}{\sin x}dx$

$$=\int\sqrt{\frac{\cos^2 x-\sin^2 x}{\sin^2 x}}\,dx=\int\sqrt{\cot^2 x-1}\,dx$$

Let $\cot x = \sec\theta \Rightarrow -\operatorname{cosec}^2 x\,dx = \sec\theta\tan\theta\,d\theta$

$$\therefore I=\int\sqrt{\sec^2\theta-1}.\frac{\sec\theta\tan\theta}{-(1+\sec^2\theta)}d\theta$$

$$=-\int\frac{\sec\theta.\tan^2\theta}{1+\sec^2\theta}d\theta=-\int\frac{\sin^2\theta}{\cos\theta+\cos^3\theta}d\theta$$

$$=-\int\frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta\ =-\int\frac{(1+\cos^2\theta)-2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta$$

$$=-\int\sec\theta d\theta+2\int\frac{\cos\theta}{1+\cos^2\theta}d\theta$$

$$=-\log|\sec\theta+\tan\theta|+2\int\frac{\cos\theta}{2-\sin^2\theta}d\theta$$

$$=-\log|\sec\theta+\tan\theta|+2.\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right|+c$$

Now $\cot x=\sec\theta\Rightarrow\tan x=\cos\theta\Rightarrow\sin\theta=\sqrt{1-\tan^2 x}$

$$\therefore\ \mathrm{I}=-\log\left|\cot x+\sqrt{\cot^2 x-1}\right|+\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right|+c$$

15. $I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Put $x = \cos^2\theta \Rightarrow dx = -2\cos\theta\sin\theta\,d\theta$

$$\therefore\quad I=-\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.2\sin\theta\cos\theta\,d\theta$$

$$=-\int\frac{\sin\theta/2}{\cos\theta/2}.2.2\sin(\theta/2)\cos(\theta/2)\cos\theta\,d\theta$$

$$=-2\int(1-\cos\theta)\cos\theta\,d\theta=-2\int(\cos\theta-\cos^2\theta)\,d\theta$$

$$=-2\int\left(\cos\theta-\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=-2\left[\sin\theta-\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+c\right]$$

$$=-2\sqrt{1-x}+\left[\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}\right]+c$$

$$=-\ 2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+c$$

16. $I=\int\frac{dx}{x^2\left(x^4+1\right)^{3/4}}=\int\frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$

Put $1+\frac{1}{x^4}=t\ \Rightarrow\ \frac{-4}{x^5}dx=dt\ \Rightarrow\ \frac{dx}{x^5}=-\frac{dt}{4}$

$$\therefore\quad I=\int\frac{-dt}{4t^{3/4}}=\frac{-1}{4}\left(\frac{t^{-3/4+1}}{\frac{-3}{4}+1}\right)+c$$

$$=-t^{1/4}+c=-\left(1+\frac{1}{x^4}\right)^{1/4}+c$$

17. $\int(e^{\log x}+\sin x)\cos x\,dx$

$$=\int(x+\sin x)\cos x\,dx\ =\int x\cos x+\frac{1}{2}\int\sin 2x\,dx$$

$$=x\sin x-\int\sin x\,dx+\frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$=x\sin x+\cos x-\frac{1}{4}\cos 2x+c$$

18. $I=\int\frac{x^2dx}{(a+bx)^2}$

Let $a+bx=t\ \Rightarrow\ x=\left(\frac{t-a}{b}\right)\ \Rightarrow\ dx=\frac{dt}{b}$

$$\therefore\quad I=\frac{1}{b^3}\int\frac{(t-a)^2}{t^2}dt=\frac{1}{b^3}\int\frac{t^2-2at+a^2}{t^2}dt$$

$$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt\ =\frac{1}{b^3}\left[t-2a\log|t|-\frac{a^2}{t}\right]+c$$

$$=\frac{1}{b^3}\left[a+bx-2a\log|a+bx|-\frac{a^2}{a+bx}\right]+c$$

19. $I\ =\int\frac{\sin x}{\sin x-\cos x}dx$

$$=\frac{1}{2}\int\frac{\sin x+\cos x+\sin x-\cos x}{\sin x-\cos x}dx$$

$$=\frac{1}{2}\int\frac{\cos x+\sin x}{\sin x-\cos x}dx+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x-\cos x}dx$$

$$=\frac{1}{2}\log|\sin x-\cos x|+\frac{x}{2}+c$$

Topic-2: Integration of the Forms: $\int e^x(f(x)+f'(x))dx$, $\int e^{kx}(df(x)+f'(x))dx$, Integration by Partial Fractions, Integration of Different Expressions of e^x

1. (d) $I=\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx=\int \frac{x^2-1}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$

$$=\frac{1}{4}\int \frac{\frac{4}{x^3}-\frac{4}{x^5}}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$$

Put $2-\frac{2}{x^2}+\frac{1}{x^4}=t \Rightarrow \left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx=dt$

$$\therefore\ I=\frac{1}{4}\int \frac{dt}{\sqrt{t}}=\frac{2\sqrt{t}}{4}+c=\frac{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}{2}+c$$

$$=\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$$

2. $I=\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx.$

$$\frac{x^3+3x+2}{(x^2+1)^2(x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

On comparing the coefficient of A, B, C, D and E from both sides and solving them, we get

$$A=-\frac{1}{2}, B=\frac{1}{2}, C=\frac{1}{2}, D=0 \text{ and } E=2.$$

$$\therefore\ I=-\frac{1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{x+1}{x^2+1}dx+2\int\frac{dx}{(x^2+1)^2}$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{1}{2}\tan^{-1}x+2I_1+c_1$$

where $I_1=\int\frac{dx}{(x^2+1)^2}$,

Now put $x=\tan\theta$

$$\therefore\ I_1=\int\frac{\sec^2\theta}{\sec^4\theta}d\theta=\int(\cos^2\theta)\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta$$

$$\Rightarrow\ I_1=\frac{1}{2}(\theta+\frac{1}{2}\sin 2\theta)=\frac{1}{2}\tan^{-1}x+\frac{1}{4}\cdot\frac{2x}{1+x^2}$$

$$\therefore\ I=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{3}{2}\tan^{-1}x$$

$+\frac{x}{1+x^2}+c$ where c is constant of integration.

3. $I=\int\frac{(x-1)e^x}{(x+1)^3}dx=\int\frac{(x+1-2)e^x}{(x+1)^3}dx$

$$=\int\left[\frac{1}{(x+1)^2}-\frac{2}{(x+1)^3}\right]e^x dx=\frac{e^x}{(x+1)^2}+c$$

$[\because \int e^x(f(x)+f'(x))dx=e^x f(x)]$

Topic-3: Evaluation of Definite Integral by Substitution, Properties of Definite Integrals

1. (c) Given that $\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt<g(x)<2\sqrt{x}$

$$\Rightarrow \int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt\ \sqrt{n}\le f(x)g(x)\le 2\sqrt{x}\ \sqrt{n}$$

Since,

$$\Rightarrow \int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt=\sin^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}-\sin^{-1}x-x\sqrt{1-x^2}$$

$$\therefore \lim_{x\to 0}\frac{\left(\sin^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}-\sin^{-1}x-x\sqrt{1-x^2}\right)}{\sqrt{x}}\le$$

$$f(x)g(x)\le\frac{2\sqrt{x}}{\sqrt{x}}$$

$$\Rightarrow 2\le\lim_{x\to 0}f(x)g(x)\le 2.$$

$$\Rightarrow \lim_{x\to 0}f(x)g(x)=2.$$

2. (b) Given that

$$f(n)=n+\frac{16+5n-3n^2}{4n+3n^2}+\frac{32+n-3n^2}{8n+3n^2}+\ldots+\frac{25n-7n^2}{7n^2}$$

$$=\left(\frac{16+5n-3n^2}{4n+3n^2}+1\right)+\left(\frac{32+n-3n^2}{8n+3n^2}+1\right)+\ldots+\left(\frac{25n-7n^2}{7n^2}+1\right)$$

$$\Rightarrow f(n)=\frac{9n+16}{4n+3n^2}+\frac{9n+32}{8n+3n^2}+\ldots\ldots+\frac{25n}{7n^2}$$

$$=\sum_{r=1}^{n}\frac{9n+16r}{4rn+3n^2}=\frac{1}{n}\sum_{r=1}^{n}\frac{9+16\left(\frac{r}{n}\right)}{4\left(\frac{r}{n}\right)+3}$$

Let $\frac{1}{n}=dx$ and $\frac{r}{n}=x$ when $r=1$ and $n\to\infty$ then $x\to 0$ when $r=n$, then $x\to 1$

$$\therefore \lim_{n\to\infty}f(n)=\int_0^1\frac{9+16x}{4x+3}dx$$

$$=\int_0^1\frac{(16x+12)-3}{4x+3}dx$$

$$=\left(4x-\frac{3}{4}\ln|4x+3|\right)\Big|_0^1=4-\frac{3}{4}\ln\frac{7}{3}$$

3. (a) $I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx$ ....(i)

$I=\int_{-\pi/2}^{\pi/2}\frac{e^x x^2\cos x}{1+e^x}dx$ ....(ii)

$\left[\because \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right]$

On adding (i) and (ii), we get

$2I=\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx=2\int_0^{\pi/2}x^2\cos x\,dx$

$I=\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\frac{\pi}{2}}=\frac{\pi^2}{4}-2$

4. (a) Let $I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\text{cosec }x)^{17}dx$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\text{cosec }x+\cot x+\text{cosec }x-\cot x)^{16}\,2\text{cosec }x\,dx$

$I=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\text{cosec }x+\cot x+\frac{1}{\text{cosec }x+\cot x}\right)^{16}.\text{cosec }x\,dx$

Let cosec x + cot $x = e^u$

$\Rightarrow(-\text{cosec }x\cot x-\text{cosec}^2 x)\,dx=e^u\,du$

$\Rightarrow-\text{cosec }x\,dx=du$

Also at $x=\frac{\pi}{4},\ u=ln\left(\sqrt{2}+1\right)$

and at $x=\frac{\pi}{2}, u=ln\ 1=0$

$\therefore I=-2\int_{ln(\sqrt{2}+1)}^{0}\left(e^u+e^{-u}\right)^{16}du=2\int_0^{ln(\sqrt{2}+1)}\left(e^u+e^{-u}\right)^{16}du$

5. (b) $F(x)=\int_0^{x^2}f\left(\sqrt{t}\right)dt$ for $x\in[0,2]$

$\Rightarrow F'(x)=f(x).2x$

Now $F'(x)=f'(x)\forall x\in(0,2)$

$\Rightarrow f(x).2x=f'(x)\Rightarrow\frac{f'(x)}{f(x)}=2x$

On integrating w.r.t. x, we get

$ln\,f(x)=x^2+c\Rightarrow f(x)=e^{x^2+c}=e^{x^2}.\,e^c$

Since $f(0)=1\Rightarrow 1=e^c,\quad\therefore\ f(x)=e^{x^2}$

Hence $F(x)=\int_0^{x^2}e^x dx=e^{x^2}-1,\ \therefore\ F(2)=e^4-1$

6. (d) We have $f'(x)-2f(x)<0$

$\Rightarrow e^{-2x}f'(x)-2e^{-2x}f(x)<0\Rightarrow\frac{d}{dx}(e^{-2x}f(x))<0$

$\Rightarrow e^{-2x}f(x)$ is strictly decreasing function on $\left[\frac{1}{2},1\right]$

$\therefore\quad e^{-2x}f(x)<e^{-1}f\left(\frac{1}{2}\right)$ or $f(x)<e^{2x-1}$

Also given that $f(x)$ is positive function so $f(x)>0$

$\therefore\quad 0<f(x)<e^{2x-1}$

$\Rightarrow\quad 0<\int_{1/2}^1 f(x)dx<\int_{1/2}^1 e^{2x-1}dx$

$\Rightarrow\quad 0<\int_{1/2}^1 f(x)dx<\left[\frac{e^{2x-1}}{2}\right]_{1/2}^1$

$\Rightarrow\quad\int_{1/2}^1 f(x)dx\in\left(0,\frac{e-1}{2}\right)$

7. (b) $\int_{-\pi/2}^{\pi/2}\left[x^2+\ln\left(\frac{\pi+x}{\pi-x}\right)\right]\cos x\,dx$

$=\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx+\int_{-\pi/2}^{\pi/2}\ln\left(\frac{\pi+x}{\pi-x}\right)\cos x\,dx$

$=2\int_0^{\pi/2}x^2\cos x\,dx+0$ [$\because x^2\cos x$ is an even function and $\ln\left(\frac{\pi+x}{\pi-x}\right)\cos x$ is an odd function]

$=2\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}=\frac{\pi^2}{2}-4$

8. (a) $I=\frac{1}{2}\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}}\frac{2x\sin x^2}{\sin x^2+\sin(\ell n6-x^2)}dx$

Let $x^2=t\Rightarrow 2x\,dx=dt$

Also, when $x=\sqrt{\ell n2}$, then, $t=\ell n2$

when $x=\sqrt{\ell n3}$, then, $t=\ell n3$

$\therefore I=\frac{1}{2}\int_{\ell n2}^{\ell n3}\frac{\sin t\,dt}{\sin t+\sin(\ell n6-t)}$ ...(i)

$\Rightarrow I=\frac{1}{2}\int_{\ell n2}^{\ell n3}\frac{\sin(\ell n6-t)}{\sin t+\sin(\ell n6-t)}dt$ ...(ii)

$\left[\because\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

On adding equations (i) and (ii), we get

$2I=\frac{1}{2}\int_{\ell n2}^{\ell n3}1dt=\frac{1}{2}(\ell n3-\ell n2)=\frac{1}{2}\ell n\frac{3}{2}\Rightarrow I=\frac{1}{4}\ell n\frac{3}{2}$

9. **(b)** $e^{-x}f(x)=2+\int_0^x \sqrt{1+t^4}\,dt\ \forall x\in(-1,1)$

On differentiating, we get

$-e^{-x}f(x)+e^{-x}f'(x)=0+\sqrt{1+x^4}$

$\Rightarrow -f(0)+f'(0)=1 \Rightarrow f'(0)=1+f(0)=1+2=3$

Now $f^{-1}(f(x))=x$

$\Rightarrow \left[\left(f^{-1}\right)'(f(x))\right]f'(x)=1$

$\Rightarrow \left(f^{-1}\right)'(f(0))f'(0)=1 \Rightarrow \left(f^{-1}\right)'(2)=\frac{1}{3}$

10. **(c)** Given that f is a non negative function defined on $[0, 1]$ and $\int_0^x \sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)\,dt,\ 0\le x\le 1$

Differentiating both sides with respect to x, we get

$\sqrt{1-[f'(x)]^2}=f(x)$

$\Rightarrow 1-[f'(x)]^2=[f(x)]^2 \Rightarrow [f'(x)]^2=1-[f(x)]^2$

$\Rightarrow \frac{d}{dx}f(x)=\pm\sqrt{1-[f(x)]^2} \Rightarrow \pm\frac{d\,f(x)}{\sqrt{1-[f(x)]^2}}=dx$

Integrating both sides with respect to x, we get

$\pm\int\frac{d\,f(x)}{\sqrt{1-[f(x)]^2}}=\int dx \Rightarrow \pm\sin^{-1}f(x)=x+C$

$\because$ Given that $f(0)=0 \Rightarrow C=0$

Hence $f(x)=\pm\sin x$

But as $f(x)$ is a non negative function on $[0, 1]$

$\therefore f(x)=\sin x.$

Now $\sin x<x, \forall x>0$

$\therefore f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}.$

11. **(c)** $I=\int_{-2}^{0}[x^3+3x^2+3x+3+(x+1)\cos(x+1)]dx$

$=\left[\frac{x^4}{4}+x^3+\frac{3x^2}{2}+3x+(x+1)\sin(x+1)+\cos(x+1)\right]_{-2}^{0}$

$=(\sin 1+\cos 1)-(4-8+6-6+\sin 1+\cos 1)=4$

12. **(c)** $\int_{\sin x}^{1} t^2 f(t)\,dt=1-\sin x$

$\Rightarrow \frac{d}{dx}\int_{\sin x}^{1} t^2 f(t)\,dt=\frac{d}{dx}(1-\sin x)$

$\Rightarrow -\sin^2 x\, f(\sin x).\cos x=-\cos x$

$\Rightarrow f(\sin x)=\frac{1}{\sin^2 x} \Rightarrow f(1/\sqrt{3})=\frac{1}{(1/\sqrt{3})^2}=3$

13. **(b)** $I=\int_0^1\sqrt{\frac{1-x}{1+x}}dx=\int_0^1\frac{1-x}{\sqrt{1-x^2}}dx$

$=\sin^{-1}x\Big|_0^1-\left(-\frac{1}{2}\right)\int_0^1\frac{2x}{\sqrt{1-x^2}}dx$

$=\frac{\pi}{2}+\frac{1}{2}\left[2\sqrt{1-x^2}\Big|_0^1\right]=\frac{\pi}{2}-1$

14. **(a)** $\int_0^{t^2} xf(x)dx=\frac{2}{5}t^5$ (Here, $t>0$)

Differentiating both sides w.r.t. t [Using Leibnitz theorem]

$\Rightarrow t^2 f(t^2)\times 2t-0=\frac{2}{5}\times 5t^4 \Rightarrow f(t^2)=t$

$\left[\because, \frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)dt=f[\psi(x)].\psi'(x)-f[\phi(x)]\,\phi'(x)\right]$

$\therefore f\left(\frac{4}{25}\right)=\frac{2}{5}$

15. **(d)** We have $f(x)=\int_{x^2}^{x^2+1}e^{-t^2}dt$

Then $f'(x)=e^{-(x^2+1)^2}.2x-e^{-x^4}.2x$

$\left[\because, \frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)dt=f[\psi(x)].\psi'(x)-f[\phi(x)]\,\phi'(x)\right]$

$\because (x^2+1)^2>x^4\ \therefore f'(x)>0, \forall x<0$

$\therefore$ $f(x)$ increases when $x<0$

16. **(a)** Let $I=\int_{-1/2}^{1/2}\left([x]+\ln\left(\frac{1+x}{1-x}\right)\right)dx$

$=\int_{-1/2}^{1/2}[x]dx+\int_{-1/2}^{1/2}\ln\left(\frac{1+x}{1-x}\right)dx$

$=\int_{-1/2}^{0}(-1)dx+0=-1/2$

$\left[\because \log\left(\frac{1+x}{1-x}\right)\text{is an odd function}\right]$

17. **(c)** Given that $T>0$ is a fixed real number. f is continuous $\forall x\in R$ such that $f(x+T)=f(x)$

$\Rightarrow f$ is a periodic function of period T

Also given $I=\int_0^T f(x)dx$

Then let $I_1=\int_3^{3+3T} f(2x)dx$

Put $2x=z \Rightarrow dx=\frac{dz}{2}$

also as $x\to 3, z\to 6$; as $x\to 3+3T, z\to 6+6T$

$$I_1=\frac{1}{2}\int_6^{6+6T} f(z)dz$$

$$=\frac{1}{2}\left[\int_6^T f(z)dz+\sum_{n=1}^{5}\int_{nT}^{(n+1)T} f(z)dz+\int_{6T}^{6T+6} f(z)dz\right]$$

Now, $\int_{nT}^{(n+1)T} f(z)dz=\int_0^T f(nT+u)du,$

where $z=nT+u$

$=\int_0^T f(u)du=\mathrm{I}$ $\quad [\because f(nT+u)=f(u)]$

Similarly, we can show that

$$\int_{6T}^{6T+6} f(z)dz=\int_0^6 f(z)dz$$

$$\therefore I_1=\frac{1}{2}\left[\int_6^T f(z)dz+5I+\int_0^6 f(z)dz\right]$$

$$=\frac{1}{2}\left[\int_0^T f(z)dz+5I\right]=\frac{1}{2}(6I)=3\mathrm{I}$$

18. **(a)** Here $f(x)=\int_1^x \sqrt{2-t^2}\,dt \Rightarrow f'(x)=\sqrt{2-x^2}$

$$\left[\because \frac{d}{dx}\int_{\psi(x)}^{\psi(x)} f(t)\,dt=f[\psi(x)]\psi'(x)-f[\phi(x)]\phi'(x)\right]$$

Hence, the given equation $x^2-f'(x)=0$ becomes

$$x^2-\sqrt{2-x^2}=0 \Rightarrow x^2=\sqrt{2-x^2} \Rightarrow x=\pm 1$$

19. **(c)** Given : $F(x)=\int_0^x f(t)dt$ and $F(x^2)=x^2(1+x)$

$\therefore$ $F'(x)=f(x)$ and $F'(x^2)\,.\,2x=2x+3x^2$

$$\Rightarrow F'(x^2)=\left(\frac{2+3x}{2}\right) \Rightarrow f(x^2)=\frac{2+3x}{2}$$

$[\because F'(x)=f(x)]$

$$\therefore f(4)=\frac{2+3\times 2}{2}=\frac{8}{2}=4$$

20. **(c)** $I=\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx$ .....(i)

Put $x=-y \Rightarrow dx=-dy$

$$I=-\int_{\pi}^{-\pi}\frac{\cos^2 y}{1+a^{-y}}dy=\int_{-\pi}^{\pi}\frac{a^y\cos^2 y}{1+a^y}dy$$

$$I=-\int_{-\pi}^{\pi}\frac{a^x\cos^2 x}{1+a^x}dx \quad \left[\because \int_a^b f(y)dy=\int_a^b f(x)dx\right] \quad \text{...(ii)}$$

On adding (i) and (ii), we get

$$2I=\int_{-\pi}^{\pi}\frac{(1+a^x)\cos^2 x}{(1+a^x)}dx=\int_{-\pi}^{\pi}\cos^2 x\,dx$$

$2I=2\int_0^{\pi}\cos^2 x\,dx$ [Even function]

$I=2\int_0^{\pi/2}\cos^2 x\,dx$ ....(iii)

$$\left[\because \int_0^{2a} f(x)dx=2\int_0^a f(x)dx \text{ if } f(2a-x)=f(x)\right]$$

$=2\int_0^{\pi/2}\sin^2 x\,dx$ ....(iv)

On adding (iii) and (iv), we get

$$2I=2\int_0^{\pi/2}(\cos^2 x+\sin^2 x)dx=2.\pi/2=\pi \therefore I=\pi/2$$

21. **(b)** Let $I=\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right|dx$

We know that for $\frac{1}{e}<x<1, \log_e x<0$ and hence

$$\frac{\log_e x}{x}<0$$

Also for $1<x<e^2$, $\log_e x>0$ and hence $\frac{\log_e x}{x}>0$

$$\therefore I=\int_{1/e}^{1}\left[-\frac{\log_e x}{x}\right]dx+\int_1^{e^2}\frac{\log_e x}{x}dx$$

$$=-\frac{1}{2}\left[(\log_e x)^2\right]_{1/e}^{1}+\frac{1}{2}\left[(\log_e x)^2\right]_1^{e^2}=\frac{1}{2}+2=\frac{5}{2}.$$

22. **(c)** If $f(x)=\begin{cases} e^{\cos x}\sin x & for\,|x|\le 2 \\ 2 & \text{otherwise}\end{cases}$

$$\Rightarrow f(x)=\begin{cases} e^{\cos x}\sin x & for\ -2\le x\le 2 \\ 2 & \text{otherwise}\end{cases}$$

$$\int_{-2}^{3} f(x)dx=\int_{-2}^{2} f(x)dx+\int_2^3 f(x)dx$$

$$= \int_{-2}^{2} e^{\cos x}\sin x dx + \int_{2}^{3} 2dx = 0 + 2[x]_2^3$$

[$\because e^{\cos x}\sin x$ is an odd function.]

$= 2[3-2] = 2$

23. (b) $g(x) = \int_0^x f(t)dt$

$$\Rightarrow g(2) = \int_0^2 f(t)dt = \int_0^1 f(t)dt + \int_1^2 f(t)dt$$

Now, $\frac{1}{2} \le f(t) \le 1$ for $t \in [0,1]$

$$\Rightarrow \int_0^1 \frac{1}{2}dt \le \int_0^1 f(t)dt \le \int_0^1 1dt$$

$$\Rightarrow \frac{1}{2} \le \int_0^1 f(t)dt \le 1 \quad \text{... (i)}$$

Again, $0 \le f(t) \le \frac{1}{2}$ for $t \in [1,2]$

$$\Rightarrow \int_1^2 0dt \le \int_1^2 f(t)dt \le \int_1^2 \frac{1}{2}dt$$

$$\Rightarrow 0 \le \int_1^2 f(t)dt \le \frac{1}{2} \quad \text{... (ii)}$$

From (i) and (ii), we get

$$\frac{1}{2} \le \int_0^1 f(t)dt + \int_1^2 f(t)dt \le \frac{3}{2} \Rightarrow \frac{1}{2} \le g(2) \le \frac{3}{2}$$

Hence, $0 \le g(2) \le 2$ is the most appropriate solution.

24. (c) $$[2\sin x] = \begin{cases} 1, & \text{if } \frac{\pi}{2} \le x < \frac{5\pi}{6} \\ 0, & \text{if } \frac{5\pi}{6} \le x < \pi \\ -1, & \text{if } \pi \le x < \frac{7\pi}{6} \\ -2, & \text{if } \frac{7\pi}{6} \le x \le \frac{3\pi}{2} \end{cases}$$

$$\therefore I = \int_{\pi/2}^{5\pi/6} 1.dx + \int_{5\pi/6}^{\pi} 0.dx + \int_{\pi}^{7\pi/6} (-1)dx + \int_{7\pi/6}^{3\pi/2} (-2)dx$$

$$= \left[\frac{5\pi}{6} - \frac{\pi}{2}\right] + 0 - 1\left[\frac{7\pi}{6} - \pi\right] - 2\left[\frac{3\pi}{2} - \frac{7\pi}{6}\right]$$

$$= \frac{2\pi}{6} - \frac{\pi}{6} - \frac{4\pi}{6} = \frac{-3\pi}{6} = \frac{-\pi}{2}$$

25. (a) $$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x} \quad \text{... (i)}$$

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(\pi - x)} \quad \left[\because \int_a^b f(x)dx = \int_a^b (f(a+b-x)dx\right]$$

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x} \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\right)dx$$

$$= \int_{\pi/4}^{3\pi/4} 2\operatorname{cosec}^2 x\, dx = -2[\cot x]_{\pi/4}^{3\pi/4}$$

$$= -2[\cot 3\pi/4 - \cot \pi/4] = -2(-1-1) = 4$$

$\therefore$ I = 2

26. (a) $g(x) = \int_0^x \cos^4 t\, dt$

$$\therefore g(x+\pi) = \int_0^{x+\pi} \cos^4 t\, dt = \int_0^{\pi} \cos^4 t\, dt + \int_{\pi}^{x+\pi} \cos^4 t\, dt$$

$g(x+\pi) = g(\pi) + I$, where $I = \int_{\pi}^{x+\pi} \cos^4 t\, dt$

Put $t = \pi + y \Rightarrow dt = dy$

Also as $t \to \pi, y \to 0$

As $t \to x+\pi, y \to x$

$$\therefore I = \int_0^x \cos^4(\pi + y)dy = \int_0^x \cos^4 y\, dy$$

$$= \int_0^x \cos^4 t\, dt = g(x)$$

Hence, $g(x+\pi) = g(\pi) + g(x)$

27. (a) Let $I = \int_{\pi}^{2\pi} [2\sin x]dx$

Now, $\pi \le x < 7\pi/6 \Rightarrow -1 \le 2\sin x < 0$

$\Rightarrow [2\sin x] = -1$

and $7\pi/6 \le x < 11\pi/6 \Rightarrow -2 \le 2\sin x < -1$

$\Rightarrow$ [2 sin x] = – 2

$\frac{11\pi}{6} \le x \le 2\pi$

–1< 2 sinx < 0

[2sinx] = –1

$$\therefore\ I=\int_{\pi}^{7\pi/6}(-1)\,dx+\int_{7\pi/6}^{11\pi/6}(-2)\,dx+\int_{11\pi/6}^{2\pi}(-1)\,dx$$

$$=\left(-\frac{7\pi}{6}+\pi\right)+2\left(-\frac{11\pi}{6}+\frac{7\pi}{6}\right)+\left(-2\pi+\frac{11\pi}{6}\right)$$

$$=-\frac{\pi}{6}-\frac{8\pi}{6}-\frac{\pi}{6}=-\frac{10\pi}{6}=\frac{-5\pi}{3}$$

28. **(d)** $f(x)=A\sin(\pi x/2)+B$

$$\Rightarrow f'(x)=\frac{A\pi}{2}\cos\left(\frac{\pi x}{2}\right)\ \Rightarrow\ f'\left(\frac{1}{2}\right)=\frac{A\pi}{2}\cos\frac{\pi}{4}=\sqrt{2}$$

$$\Rightarrow A=4/\pi \text{ and } \int_0^1 f(x)dx=\frac{2A}{\pi}$$

$$\Rightarrow \int_0^1\left[A\sin\left(\frac{\pi x}{2}\right)+B\right]dx=\frac{2A}{\pi}$$

$$\Rightarrow \left|-\frac{2A}{\pi}\cos\left(\frac{\pi x}{2}\right)+Bx\right|_0^1=\frac{2A}{\pi}$$

$$\therefore\ B+\frac{2A}{\pi}=\frac{2A}{\pi}\Rightarrow B=0$$

29. **(d)** Let $I=\int_0^{\frac{\pi}{2}}\frac{dx}{1+\tan^3 x}=\int_0^{\frac{\pi}{2}}\frac{\cos^3 x}{\sin^3 x+\cos^3 x}dx$ ... (i)

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^3\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right)+\cos^3\left(\frac{\pi}{2}-x\right)}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^3 x}{\cos^3 x+\sin^3 x}dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I=\int_0^{\frac{\pi}{2}}\frac{\cos^3 x+\sin^3 x}{\sin^3 x+\cos^3 x}dx=\int_0^{\frac{\pi}{2}}1dx=\frac{\pi}{2};\ \therefore\ I=\frac{\pi}{4}$$

30. **(d)** $I=\int_{-\pi/2}^{\pi/2}\{f(x)+f(-x)\}\{g(x)-g(-x)\}dx$

Let $F(x)=(f(x)+f(-x))(g(x)-g(-x))$

then $F(-x)=(f(-x)+f(x))(g(-x)-g(x))$

$$=-[f(x)+f(-x)][g(x)-g(-x)]\ =-F(x)$$

Hence, $F(x)$ is an odd function, $\therefore\ I=0$

31. **(c)** $I=\int_0^{\pi}e^{\cos^2 x}\cos^3(2n+1)x\,dx, n\in Z$ ... (i)

$$=\int_0^{\pi}e^{\cos^2(\pi-x)}\cos^3[(2n+1)(\pi-x)]dx$$

$$\because\ \int_0^a f(x)dx=\int_0^a f(a-x)dx$$

$$\therefore\ I=\int_0^{\pi}e^{\cos^2 x}\cos^3[(2n+1)\pi-(2n+1)x]dx$$

$$I=\int_0^{\pi}(-e^{\cos^2 x}\cos^3(2n+1)x)\,dx \quad \text{...(ii)}$$

On adding (i) and (ii), we get

$2I=0\Rightarrow I=0$

32. **(a)** $I=\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$ ...(i)

$$=\int_0^{\pi/2}\frac{\sqrt{\cot(\pi/2-x)}}{\sqrt{\cot(\pi/2-x)}+\sqrt{\tan(\pi/2-x)}}dx$$

$$I=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}=dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I=\int_0^{\pi/2}\frac{\sqrt{\cot x}+\sqrt{\tan x}}{\sqrt{\cot x}+\sqrt{\tan x}}=dx=\int_0^{\pi/2}1.dx$$

$=[x]_0^{\pi/2}=\pi/2$, $\therefore$

$I=\pi/4$

33. **(b)** $\int_0^1(1+\cos^8 x)(ax^2+bx+c)\,dx$

$$=\int_0^2(1+\cos^8 x)(ax^2+bx+c)\,dx$$

$$=\int_0^1(1+\cos^8 x)(ax^2+bx+c)\,dx+\int_1^2(1+\cos^8 x)(ax^2+bx+c)\,dx$$

$$\Rightarrow \int_1^2(1+\cos^8 x)(ax^2+bx+c)\,dx=0$$

We know that if $\int_\alpha^\beta f(x)dx=0$, then $f(x)$ is + ve on some part of (α, β) and – ve on other part of (α, β).

But here $1+\cos^8 x$ is always + ve,

$\therefore$ ax^2+bx+c is + ve on some part of [1 , 2] and – ve on other part [1, 2]

$\therefore$ $ax^2+bx+c=0$ has at least one root in (1, 2).

Hence, $ax^2+bx+c=0$ has at least one root in (0, 2).

34. **(d)**

$$\int_0^1(1+e^{-x^2})dx=\int_0^1\left(1+1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots\infty\right)dx$$

$$=\left[2x-\frac{x^3}{3.1!}+\frac{x^5}{5.2!}-\frac{x^7}{7.3!}+\ldots\infty\right]_0^1$$

$$=\left[2-\frac{1}{3.1!}+\frac{1}{5.2!}-\frac{1}{7.3!}+\ldots\infty\right]$$

35. **(5)** $\because\ I=\int_1^2 \log_2(x^3+1)\,dx+\int_1^{\log_2 9}(2^x-1)^{\frac{1}{3}}\,dx$

Let $I_2=\int_1^{\log_2 9}(2^x-1)^{\frac{1}{3}}\,dx$

Put $2^x-1=t^3 \Rightarrow 2^x\ln 2\,dx=3t^2dt$

$$\Rightarrow\quad dx=\frac{3t^2}{\ln 2(t^3+1)}dt$$

when $x=1$, $t=1$ when $x=\log_2 9$, $t=2$

$$\therefore\quad I_2=\int_1^2\frac{3t^3}{(t^3+1)\ln 2}dt$$

$$\text{So, } I=\int_1^2 \log_2(x^3+1)dx+\int_1^2\frac{3t^3}{\ln 2(t^3+1)}dt$$

$$\Rightarrow\quad I=\int_1^2\left(\log_2\left(t^3+1\right)+t\cdot\frac{3t^2}{(t^3+1)\ln 2}\right)dt$$

$$\left[\because \int_a^b f(x)dx=\int_a^b f(t)dt\right]$$

$$=\int_1^2 d(t.\log_2(t^3+1))$$

$$=t.\log_2(t^3+1)\Big|_1^2$$

$= 2\log_2 9-1\log_2 2$

$= 2\log_2 9-1$

$\because\quad 8<9<2^{7/2}\Rightarrow 3<\log_2 9<\dfrac{7}{2}$

$\Rightarrow\quad 5<2\log_2 9-1<6$

So, $[I]=5$

36. **(182)** Let $y=\dfrac{10x}{x+1}$, $\quad 0\le x\le 10$

$\Rightarrow\quad xy+y=10x$

$\Rightarrow\quad x=\dfrac{y}{10-y}$

$\Rightarrow\quad 0\le\dfrac{y}{10-y}\le 10$

$\Rightarrow\quad \dfrac{y}{y-10}\le 0$ and $\dfrac{y}{10-y}\le 10$

$$\Rightarrow\quad \frac{11y-100}{y-10}\ge 0$$

$\Rightarrow$ $+\ -\ +$ on $0,\ 10$; $\quad +\ -\ +$ on $\dfrac{100}{11},\ 10$

$y\in[0,10)$ and $y\in\left(-\infty,\dfrac{100}{11}\right]U(10,\infty)$

$\Rightarrow\quad y\in\left[0,\dfrac{100}{11}\right]$

$\sqrt{y}\in\left[0,\dfrac{10}{\sqrt{11}}\right]\quad\Rightarrow\quad\left[\sqrt{y}\right]=\{0,1,2,3\}.$

Case I : $0\le\dfrac{10x}{x+1}<1$

$\therefore\quad x\in\left[0,\dfrac{1}{9}\right)$ then $\left[\sqrt{\dfrac{10x}{x+1}}\right]=0$

Case II : $1\le\dfrac{10x}{x+1}<4$

$\therefore\quad x\in\left[\dfrac{1}{9},\dfrac{2}{3}\right)$ then $\left[\sqrt{\dfrac{10x}{x+1}}\right]=1$

Case III : $4<\dfrac{10x}{x+1}<9$

$\therefore\quad x\in\left[\dfrac{2}{3},9\right)$ then $\left[\sqrt{\dfrac{10x}{x+1}}\right]=2$

Case IV : $x\in[9,10)$

$\Rightarrow\quad$ then $\left[\sqrt{\dfrac{10x}{x+1}}\right]=3$

$$\therefore\quad I=\int_0^{1/9}0.dx+\int_{1/9}^{2/3}1.dx+\int_{2/3}^{9}2.dx+\int_9^{10}3.dx$$

$$=\left(\frac{2}{3}-\frac{1}{9}\right)+2\left(9-\frac{2}{3}\right)+3(10-9)$$

$$=\frac{5}{9}+\frac{50}{3}+3=\frac{182}{9}$$

$\therefore\quad 9I=182.$

37. **(4.00)** $F(x)=\displaystyle\int_0^x f(t).dt$

$\Rightarrow F'(x)=f(x)$

$$I=\int_0^\pi f'(x).\cos x\,dx+\int_0^\pi F(x)\cos x\,dx=2 \qquad ...(i)$$

$$I=\int_0^\pi f'(x).\cos x\,dx+\int_0^\pi F(x)\cos x\,dx=2$$

Using by parts

$$\Rightarrow I=(\cos x.f(x))_0^\pi+\int_0^\pi \sin x.f(x)dx+\int_0^\pi F(x)\cos x\,dx=2$$

$$\Rightarrow I=6-f(0)+\int_0^\pi \sin x.F'(x)dx+\int_0^\pi F(x)\cos x\,dx=2$$

$$I = 6 - f(0) + \int_0^{\pi} F'(x)\sin x dx + \int_0^{\pi} F(x)\cos x dx = 2$$

$$\Rightarrow I = 6 - f(0) + [\sin x F(x)]_0^{\pi} - \int_0^{\pi} F(x)\cos x dx + \int_0^{\pi} F(x)\cos x dx = 2$$

$$\Rightarrow I = 6 - f(0) + 0 = 2$$

$$\Rightarrow f(0) = 4$$

38. **(4)** $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ ...(i)

$$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{-\sin x})(2-\cos 2x)}$$

$$\left[\text{using} \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{e^{\sin x}}{(e^{\sin x}+1)(2-\cos 2x)}$ ...(ii)

Adding (i) and (ii):

$$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{1+e^{\sin x}}{(e^{\sin x}+1)(2-\cos 2x)}dx$$

$$= \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}dx$$

$$I = \frac{2}{\pi}\int_0^{\pi/4} \frac{dx}{1+2\sin^2 x}$$

$$= \frac{2}{\pi}\int_0^{\pi/4} \frac{\sec^2 x}{\sec^2 x + 2\tan^2 x}dx = \frac{2}{\pi}\int_0^{\pi/4} \frac{\sec^2 x}{1+3\tan^2 x}dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

At $x = 0, t = 0$; At $x = \frac{\pi}{4}, t = 1$

$$\therefore I = \frac{2}{\pi}\int_0^1 \frac{1}{1+3t^2}dt = \frac{2}{\pi}\left[\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}t)\right]_0^1$$

$$= \frac{2}{\pi}\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{3}\right] = \frac{2}{3\sqrt{3}}$$

$$\therefore 27 I^2 = 4$$

39. **(2)** Let $I = \int_0^{\frac{1}{2}} \frac{(1+\sqrt{3})dx}{\left[(1+x)^2 (1-x)^6\right]^{1/4}}$

$$= \int_0^{\frac{1}{2}} \frac{(1+\sqrt{3})dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$$

Now put $\frac{1-x}{1+x} = t \Rightarrow \frac{-2dx}{(1+x)^2} = dt$

$$\therefore I = \int_1^{1/3} \frac{(1+\sqrt{3})dt}{-2t^{6/4}} = \frac{-(1+\sqrt{3})}{2} \times \left|\frac{-2}{\sqrt{t}}\right|_1^{1/3}$$

$$= (1+\sqrt{3})(\sqrt{3}-1) = 2$$

40. **(2)** Given $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$, $f'(0) = 1$

$$g(x) = \int_x^{\frac{\pi}{2}} [f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]dt$$

$$g(x) = \int_x^{\frac{\pi}{2}} \frac{d}{dt}(f(t)\operatorname{cosec} t)\, dt$$

$$= f\left(\frac{\pi}{2}\right)\operatorname{cosec}\frac{\pi}{2} - f(x)\operatorname{cosec} x$$

$$= 3 - f(x)\operatorname{cosec} x = 3 - \frac{f(x)}{\sin x}$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\left(3 - \frac{f(x)}{\sin x}\right) = 3 - \lim_{x\to 0}\frac{f(x)}{\sin x}$$

$$= 3 - \lim_{x\to 0}\frac{f'(x)}{\cos x} = 3 - f'(0) = 3 - 1 = 2$$

41. **(1)** Let $f(x) = \int_0^x \frac{t^2}{1+t^4}dt - 2x + 1$

$$\Rightarrow f'(x) = \frac{x^2}{1+x^4} - 2 < 0, \forall x \in [0,1]$$

$\therefore$ f is decreasing on [0, 1]

Also $f(0) = 1$

and $f(1) = \int_0^1 \frac{t^2}{1+t^4}dt - 1$

For $0 \le t \le 1 \Rightarrow 0 \le \frac{t^2}{1+t^4} < \frac{1}{2}$

$$\therefore \int_0^1 \frac{t^2}{1+t^4}dt < \frac{1}{2}$$

$\Rightarrow f(1) < 0$

$\therefore$ f (x) crosses x-axis exactly once in [0, 1]

$\therefore$ f (x) = 0 has exactly one root in [0, 1]

42. **(9)** $\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Let $9x + 3\tan^{-1}x = t \Rightarrow \frac{12+9x^2}{1+x^2}dx = dt$

$\therefore \quad \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$

$\therefore \quad \log_e \left| 1 + e^{9+\frac{3\pi}{4}} - 1 \right| - \frac{3\pi}{4} = 9$

43. **(0)** $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$

$-1 < x < 2 \Rightarrow 0 < x^2 < 4$

Also $0 < x^2 < 1 \Rightarrow f(x^2) = [x^2] = 0$

$1 \le x^2 < 2 \Rightarrow f(x^2) = [x^2] = 1$

$2 \le x^2 < 3 \Rightarrow f(x^2) = 0$ (using definition of f)

$3 \le x^2 < 4 \Rightarrow f(x^2) = 0$ (using definition of f)

Also $1 \le x^2 < 2 \Rightarrow 1 \le x < \sqrt{2}$

$\Rightarrow \quad 2 \le x + 1 < \sqrt{2} + 1$

$\Rightarrow \quad f(x+1) = 0$

$\therefore \quad I = \int_1^{\sqrt{2}} \frac{x \times 1}{2+0} dx = \left[\frac{x^2}{4} \right]_1^{\sqrt{2}} = \frac{1}{4}$

$\Rightarrow \quad 4I = 1$ or $4I - 1 = 0$

44. **(2)** $\int_0^1 4x^3 \left[\frac{d^2}{dx^2} (1-x^2)^5 \right] dx$

$= 4x^3 \left[\frac{d}{dx} (1-x^2)^5 \right] \Big|_0^1 - \int_0^1 \left[\frac{d}{dx} (1-x^2)^5 \right] .12x^2 dx$

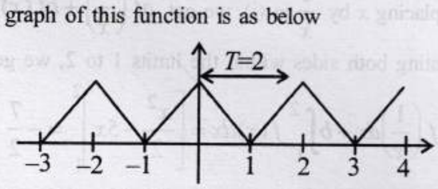
$= -12x^2 (1-x^2)^5 \Big|_0^1 + \int_0^1 (1-x^2)^5 .24x dx$

$= -12 \int_0^1 (1-x^2)^5 .(-2x) dx$

$= -12 \left(\frac{(1-x^2)^6}{6} \right)_0^1 = 2$

45. **(4)** Given function is $f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$

The graph of this function is as below



Clearly $f(x)$ is periodic with period 2

Also $\cos \pi x$ is periodic with period 2

$\therefore \ f(x) \cos \pi x$ is periodic with period 2

$\therefore \ I = \frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx$

$= \frac{\pi^2}{10} \times 10 \int_0^2 f(x) \cos \pi x \, dx$

$= \pi^2 \left[\int_0^1 (1-x) \cos \pi x \, dx + \int_1^2 (x-1) \cos \pi x \, dx \right]$

$= \pi^2 \left[\left\{ (1-x) \frac{\sin \pi x}{\pi} \Big|_0^1 + \int_0^1 \frac{\sin \pi x}{\pi} dx \right\} + \left\{ (x-1) \frac{\sin \pi x}{\pi} \Big|_1^2 - \int_1^2 \frac{\sin \pi x}{\pi} dx \right\} \right]$

$= \pi^2 \left[\left(-\frac{1}{\pi^2} \cos \pi x \right) \Big|_0^1 - \left(-\frac{1}{\pi^2} \cos \pi x \right) \Big|_1^2 \right]$

$= \left[(-\cos \pi + \cos 0) - (-\cos 2\pi + \cos \pi) \right] = 4$

46. **(0)** Given that $f(x) = \int_0^x f(t) dt$

Clearly $f(0) = 0$. Also $f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1$

Integrating both sides with respect to x, we get

$\int \frac{f'(x)}{f(x)} dx = \int 1 \, dx$

$\Rightarrow \ \ln f(x) = x + \ln C \quad \Rightarrow f(x) = Ce^x$

Now $f(0) = 0 \ \Rightarrow \ Ce^0 = 0 \ \Rightarrow \ C = 0$

$\therefore \quad f(x) = 0 \ \forall \ x \Rightarrow f(\ln 5) = 0$

47. **(2.00)** $S_1 = \int_{\pi/8}^{3\pi/8} f(x) \, dx \equiv \int_{\pi/8}^{3\pi/8} \sin^2 x dx$ ...(i)

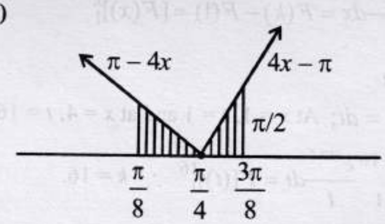
$= \int_{\pi/8}^{3\pi/8} \sin^2 \left(\frac{\pi}{8} + \frac{3\pi}{8} - x \right) dx = \int_{\pi/8}^{3\pi/8} \cos^2 x dx$ ...(ii)

Adding (i) and (ii) we get

$2S_1 = \int_{\pi/8}^{3\pi/8} (\sin^2 x + \cos^2 x) \, dx = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{\pi}{4}$

$\Rightarrow \quad \frac{16 S_1}{\pi} = 2$

48. **(1.50)**



$S_2 = \int_{\pi/8}^{3\pi/8} f(x) g_2(x) dx$

$= \int_{\pi/8}^{3\pi/8} \sin^2 x \, |4x - \pi| \, dx$ ...(i)

$= \int_{\pi/8}^{3\pi/8} \sin^2 \left(\frac{\pi}{8} + \frac{3\pi}{8} - x \right) \left| 4 \left(\frac{\pi}{8} + \frac{3\pi}{8} - x \right) - \pi \right| dx$

$= \int_{\pi/8}^{3\pi/8} (\cos^2 x) |\pi - 4x| dx$

$= \int_{\pi/8}^{3\pi/8} \cos^2 x|4x-\pi|dx$ . . . (ii)

Adding equations (i) and (ii) we get

$$2S_2 = \int_{\pi/8}^{3\pi/8} |4x-\pi|(\sin^2 x+\cos^2 x)dx = \int_{\pi/8}^{3\pi/8} |4x-\pi|dx$$

$$= 2\times\frac{1}{2}\times\frac{\pi}{8}\times\frac{\pi}{2} = \frac{\pi^2}{16}$$

$$\Rightarrow \frac{48S_2}{\pi^2} = \frac{3}{2} = 1.5$$

49. **(0.50)** $I = \int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ ...(i)

$I = \int_0^{\pi/2} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\sin\theta}+\sqrt{\cos\theta}\right)^5} d\theta$ ...(ii)

Adding two values of I, we get:

$$\frac{2}{3}I = \int_0^{\pi/2} \frac{1}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^4} d\theta$$

$$\frac{2}{3}I = \int_0^{\pi/2} \frac{\sec^2\theta}{\left(1+\sqrt{\tan\theta}\right)^4} d\theta$$

(put $\tan\theta = t^2 \Rightarrow \sec^2\theta\, d\theta = 2t\, dt$)

$$\Rightarrow \frac{2I}{3} = \int_0^{\infty} \frac{2t\,dt}{(1+t)^4}$$

$$\Rightarrow I = 3\int_0^{\infty} \frac{t+1-1}{(t+1)^4} dt = 3\int_0^{\infty} \left(\frac{1}{(t+1)^3} - \frac{1}{(t+1)^4}\right) dt$$

$$= 3\left[-\frac{1}{2(t+1)^2} + \frac{1}{3(t+1)^3}\right]_0^{\infty} = 3\times\frac{1}{6}$$

$$\Rightarrow I = \frac{1}{2} = 0.50$$

50. $\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1) = [F(x)]_1^k$

Put $x^2 = t$

$\Rightarrow$ $2xdx = dt$; At $x = 1, t = 1$ and at $x = 4, t = 16$

$\therefore$ $I = \int_1^{16} \frac{e^{\sin t}}{t} dt = F[(t)]_1^{16}$ $\therefore$ $k = 16.$

51. Let $I = \int_1^{e^{37}} \frac{\pi\sin(\pi\ln x)}{x} dx$ and $\pi\ln x = t$

$\Rightarrow$ $\frac{\pi}{x}dx = dt$, Also as $x\to 1, t\to 0, x\to e^{37}, t\to 37\pi$

$$\therefore\ I = \int_0^{37\pi} \sin t\, dt = [-\cos t]_0^{37\pi} = -\cos 37\pi + 1$$

$$= -(-1) + 1 = 2$$

52. Let $I = \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$ ...(i)

$$\Rightarrow I = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}(2\pi-x)}{\sin^{2n}(2\pi-x)+\cos^{2n}(2\pi-x)} dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$I = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$ ...(ii)

On adding (i) and (ii), we get

$$2I = \int_0^{2\pi} \frac{2\pi\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx \Rightarrow I = \pi\int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$$

$$\Rightarrow I = 2\pi\int_0^{\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a (x)dx \text{ if } f(2a-x) = f(x)\right]$$

$\Rightarrow I = 4\pi\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$ ... (iii)

[Using above property again]

$\Rightarrow I = 4\pi\int_0^{\pi/2} \frac{\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x} dx$ ....(iv)

$$\left[\because \int_0^a f(x)dx = \int_0^a (a-x)dx\right]$$

On adding (iii) and (iv), we get

$$2I = 4\pi\int_0^{\pi/2} 1.dx = 4\pi\left(\frac{\pi}{2}-0\right) = 2\pi^2 \Rightarrow I = \pi^2$$

53. $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ ...(i)

Integrating both sides within the limits 1 to 2, we get

$a\int_1^2 f(x)dx + b\int_1^2 f\left(\frac{1}{x}\right)dx = [\log x - 5x]_1^2 = \log 2 - 5$ ...(ii)

On replacing x by $\frac{1}{x}$ in (i), we get $af\left(\frac{1}{x}\right) + bf(x) = x - 5$

Integrating both sides within the limits 1 to 2, we get

$a\int_1^2 f\left(\frac{1}{x}\right)dx + b\int_1^2 f(x)dx = \left[\frac{x^2}{2} - 5x\right]_1^2 = -\frac{7}{2}$ ... (iii)

Eliminate $\int_1^2 f\left(\frac{1}{x}\right)$ between (ii) and (iii) by multiplying (ii) by a and (iii) by b and subtracting

$$\therefore\ (a^2-b^2)\int_1^2 f(x)dx = a(\log 2 - 5) + b.\frac{7}{2}$$

$$\therefore\ \int_1^2 f(x)dx = \frac{1}{(a^2-b^2)}\left[a(\log 2 - 5) + \frac{7b}{2}\right]$$

54. $I=\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ ...(i)

$I=\int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$ ...(ii)

$\left[\text{Using} \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

On adding (i) and (ii), we get

$2I=\int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$

$\Rightarrow I=\frac{1}{2}\int_2^3 1dx=\frac{1}{2}(3-2)=\frac{1}{2}$

55. $I=\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi}d\phi$ ...(i)

$\Rightarrow I=\int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin(\pi-\phi)}d\phi$

$\left[\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

$\Rightarrow I=\int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin\phi}d\phi$ ... (ii)

On adding (i) and (ii), we get $2I=\int_{\pi/4}^{3\pi/4} \frac{\pi}{1+\sin\phi}d\phi$

$=\pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{1-\sin^2\phi}d\phi=\pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{\cos^2\phi}d\phi$

$=\pi\int_{\pi/4}^{3\pi/4} (\sec^2\phi-\sec\phi\tan\phi)d\phi$

$=\pi\left[\tan\phi-\sec\phi\right]_{\pi/4}^{3\pi/4}$

$=\pi\left[\tan 3\pi/4-\sec 3\pi/4-\tan\pi/4+\sec\pi/4\right]$

$=2\pi(\sqrt{2}-1) \Rightarrow I=\pi(\sqrt{2}-1)$

56. $I=\int_{-2}^{2} |1-x^2| dx=2\int_0^2 |1-x^2| dx$

$\left[\because \int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx, \text{if } f \text{ is an even function}\right]$

$=2\int_0^1 (1-x^2)dx+2\int_1^2 (x^2-1)dx$

$=2\left[x-\frac{x^3}{3}\right]_0^1+2\left[\frac{x^3}{3}-x\right]_1^2=\frac{4}{3}+\frac{8}{3}=\frac{12}{3}=4$

57. $I=\int_0^{1.5} [x^2]dx$

We have $0<x<1.5 \Rightarrow 0<x^2<2.25$

$\therefore [x^2]=\begin{cases}0, & 0<x^2<1\\ 1, & 1\le x^2<2\\ 2, & 2\le x^2<(1.5)^2\end{cases}$

or $[x^2]=\begin{cases}0, & 0<x<1\\ 1, & 1\le x<\sqrt{2}\\ 2, & \sqrt{2}\le x<1.5\end{cases}$

$\therefore I=\int_0^{1.5}[x^2]dx=\int_0^1 0dx+\int_1^{\sqrt{2}} 1dx+\int_{\sqrt{2}}^{1.5} 2dx$

$=0+[x]_1^{\sqrt{2}}+[2x]_{\sqrt{2}}^{1.5}=\sqrt{2}-1+3-2\sqrt{2}=2-\sqrt{2}$

58. $f(x)=\begin{vmatrix}\sec x & \cos x & \sec^2 x+\cot x\operatorname{cosec} x\\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x\\ 1 & \cos^2 x & \cos^2 x\end{vmatrix}$

$=\begin{vmatrix}0 & 0 & \sec^2 x+\cot x\operatorname{cosec} x-\cos x\\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x\\ 1 & \cos^2 x & \cos^2 x\end{vmatrix}$

$[R_1\to R_1-\sec x.R_3]$

On expanding along R_1, we get

$f(x)=(\sec^2 x+\cot x\operatorname{cosec} x-\cos x)(\cos^4 x-\cos^2 x)$

$=\left(\frac{1}{\cos^2 x}+\frac{\cos x}{\sin^2 x}-\cos x\right)\cos^2 x(\cos^2 x-1)$

$=-\sin^2 x-\cos^5 x$

$\therefore \int_0^{\pi/2} f(x)dx=-\int_0^{\pi/2}(\sin^2 x+\cos^5 x)dx$

$\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx=\frac{(n-1)(n-3)\ldots 2 \, or \, 1}{(n)(n-2)\ldots 2}$

Multiply the above by $\pi/2$ when n is even

$\therefore \int_0^{\pi/2} f(x)dx=-\left[\frac{1}{2}.\frac{\pi}{2}+\frac{4}{5}.\frac{2}{3}\right]=-\left[\frac{\pi}{4}+\frac{8}{15}\right]$

$=-\left(\frac{15\pi+32}{60}\right)$

59. **(True)** Let $I=\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx$ ... (i)

$=\int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f[2a-(2a-x)]}dx$

$[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx]$

$I=\int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f(x)}$ ... (ii)

On adding (i) and (ii), we get

$2I=\int_0^{2a}\frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}dx=\int_0^{2a}1.dx$

$=[x]_0^{2a}=2a \Rightarrow I=a$

Hence, the given statement is true.

60. **(c, d)** We have, $\int_1^e \frac{(\log_e x)^{1/2}}{x(a-(\log_e x)^{3/2})^2}dx$

Let $I = \int_1^e \frac{(\ln x)^{1/2}\,dx}{x(a-(\ln x)^{3/2})^2}$

Put $a - (\ln x)^{3/2} = t$

$\Rightarrow -\frac{3}{2}(\ln x)^{1/2}\cdot\frac{1}{x}dx = dt$

$\therefore \quad I = \int_a^{a-1}\frac{\left(-\frac{2}{3}\right)dt}{t^2} \quad \left\{\because \begin{array}{l} at\ x=1, t=a \\ \text{and } at\ x=e, t=a-1\end{array}\right\}$

$= \left(-\frac{2}{3}\right)\left[\frac{t^{-1}}{-1}\right]_a^{a-1} = \frac{2}{3}\left(\frac{1}{a-1}-\frac{1}{a}\right)$

$\therefore \quad I = \left(\frac{2}{3}\right)\frac{1}{a(a-1)} = 1$

$\Rightarrow \quad 2 = 3a^2 - 3a \Rightarrow 3a^2 - 3a - 2 = 0$

$\Rightarrow \quad a = \frac{3 \pm \sqrt{9-4(3)(-2)}}{6}$

$a = \frac{3+\sqrt{33}}{6}, \frac{3-\sqrt{33}}{6}$

61. **(a,b,c)**

(a) Let $g(x) = \int_0^{\pi/3}(x)\,dx - 3\int_0^{\pi/3}\cos 3x\,dx = 0$

$g(x)$ is continuous and differentiable function and $g(0) = 0$, $g\left(\frac{\pi}{3}\right) = 0$

By Rolle's theorem

$g'(x) = f(x) - 3\cos 3x = 0$ has least one solution in $\left(0, \frac{\pi}{3}\right)$

(b) Let $h(x) = f(x) - 3\sin 3x + \frac{6}{\pi}$

Let $h(x) = \int_0^{\pi/3} f(x)\,dx - 3\int_0^{\pi/3}\sin 3x\,dx + \int_0^{\pi/3}\frac{6}{\pi}dx$

$= 0 - 2 + 2 = 0$

$h(x)$ is continuous and differentiable function and $h(0) = 0$ and $h\left(\frac{\pi}{3}\right) = 0$

By Rolle's theorem

$h'(x) = f(x) - 3\sin 3x + \frac{6}{n} = 0$ has least one solution in $\left(0, \frac{\pi}{3}\right)$

(c) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)\,dt}{1-e^{x^2}} = \lim\limits_{x\to 0}\left(\frac{x^2}{1-e^{x^2}}\right)\frac{\int_0^x f(t)\,dt}{x}$

By L'Hospital's Rule

$= -1\lim\limits_{x\to 0}\frac{f(x)}{1} = -1 \qquad [\because f(0) = 1]$

(d) $\lim\limits_{x\to 0}\frac{(\sin x)\int_0^x f(t)\,dt}{x^2} = \lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)\frac{\int_0^x f(t)\,dt}{x}$

By L'Hospital's Rule

$= 1\lim\limits_{x\to 0}\frac{f(x)}{1} = 1 \qquad [\because f(0) = 0]$

62. **(a,b,d)** (a) We know that,

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ and, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$\because \cos x \ge 1 - \frac{x^2}{2}$

$\therefore \int_0^1 x\cos x \ge \int_0^1 x\left(1-\frac{x^2}{2}\right) = \frac{1}{2}-\frac{1}{8} \Rightarrow \int_0^1 x\cos x \ge \frac{3}{8}$

(b) $\because \sin x \ge x - \frac{x^3}{6}$

$\therefore \int_0^1 x\sin x \ge \int_0^1 x\left(x-\frac{x^3}{6}\right)dx$

$\Rightarrow \int_0^1 x\sin x \ge \frac{1}{3}-\frac{1}{30} \Rightarrow \int_0^1 x\sin x\,dx \ge \frac{3}{10}$

(c) $\because \cos x < 1 \Rightarrow x^2\cos x < x^2$

$\therefore \int_0^1 x^2\cos x\,dx < \int_0^1 x^2\,dx \Rightarrow \int_0^1 x^2\cos x\,dx < \frac{1}{3}$

(d) $\int_0^1 x^2\sin x\,dx \ge \int_0^1 x^2\left(x-\frac{x^3}{6}\right)dx$

$\Rightarrow \int_0^1 x^2\sin x\,dx \ge \frac{1}{4}-\frac{1}{36} \Rightarrow \int_0^1 x^2\sin x\,dx \ge \frac{2}{9}$

63. **(a, b, d)** $F(x) = (x-1)(x-2)(x-5)$

$F(x) = \int_0^x f(t)\,dt, x > 0$

$\Rightarrow \quad F'(x) = f(x) = (x-1)(x-2)(x-5)$

Put $F'(x) = 0 \Rightarrow x = 1, 2, 5$

$F''(x) = (x-2)(x-5)+(x-1)(x-5)+(x-1)(x-2)$

$F''(1) = +$ ve, $F''(2) = -$ve, $F''(5) = +$ve

$\therefore$ F(x) has local minima at x = 1 and x = 5 and local maxima at x = 2.

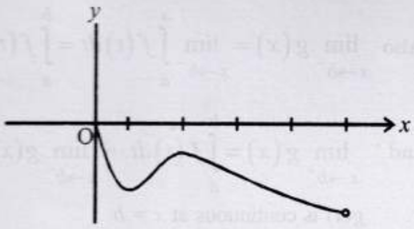
Also $F(x) = \int_0^x (t-1)(t-2)(t-5)dt$

$= \int_0^x (t^3 - 8t^2 + 17t - 10)dt = \frac{x^4}{4} - \frac{8x^3}{3} + \frac{17x^2}{2} - 10x$

$= \frac{x}{12}(3x^3 - 32x^2 + 102x - 120)$

F(1), F(2), F(5) < 0

Hence, approximate graph of F(x) for $x \in (0, 5)$ is



$\therefore$ F(x) < 0, for all $x \in (0,5)$

Thus, options (a), (b) and (d) are correct but (c) is incorrect.

64. **(b, d)** $I = \sum_{k=1}^{98}\int_k^{k+1}\frac{(k+1)}{x(x+1)}dx$; $I = \sum_{k=1}^{98}\int_k^{k+1}\frac{(k+1)}{x(x+1)^2}dx$

$\because \int_k^{k+1}\frac{1}{k+2}dx \le \int_k^{k+1}\frac{k+1}{x(x+1)}dx \le \int_k^{k+1}\frac{1}{k}dx$

$\Rightarrow I > \sum_{k=1}^{98}\frac{1}{k+2}$

$\Rightarrow I > \frac{1}{3} + ... + \frac{1}{100} > \frac{98}{100} \Rightarrow I > \frac{49}{50}$

Also, $I > \sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx = \sum_{k=1}^{98}[\log_e(k+1) - \log_e k]$

$I < \log_e 99$

65. **(Bonus)** $g(x) = \int_{\sin x}^{\sin 2x}\sin^{-1}(t)\,dt$

$\Rightarrow g'(x) = \sin^{-1}(\sin 2x)\cdot 2\cos 2x - \sin^{-1}(\sin x)\cdot \cos x$

$g'\left(\frac{\pi}{2}\right) = \sin^{-1}(\sin\pi)\cdot 2\cos\pi - \sin^{-1}\left(\sin\frac{\pi}{2}\right)\cos\frac{\pi}{2} = 0$

$g'\left(-\frac{\pi}{2}\right) = \sin^{-1}(\sin(-\pi))\cdot 2\cos(-\pi) - \sin^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right)\cdot\cos\left(\frac{-\pi}{2}\right) = 0$

66. **(a, b)** Let us check the given options one by one.

(a) Let $g(x) = x^9 - f(x)$

$\Rightarrow g(0) = -f(0) < 0$ $\quad [\because f(x) \in (0, 1)]$

And $g(1) = 1 - f(1) > 0$

$\therefore x^9 - f(x) = 0$ for some $x \in (0, 1)$

(b) Let $h(x) = x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

$h(0) = -\int_0^{\frac{\pi}{2}} f(t)\cos t\,dt < 0$

and $h(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t)\cos t\,dt > 0$

$\therefore h(0) < 0$ and $h(1) > 0$

$\Rightarrow h(x) = 0$ at some $x \in (0,1)$

$\therefore h(x) = x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt = 0$

at some $x \in (0, 1)$

(c) $e^x - \int_0^x f(t)\sin t\,dt$

$\because x \in (0, 1) \Rightarrow e^x \in (1, e)$

and $0 < f(t) < 1$ and $0 < \sin t < 1$, $\forall\ x \in (0, 1)$

$\therefore 0 < \int_0^x f(t)\sin t\,dt < 1$

$\therefore e^x - \int_0^x f(t)\sin t\,dt \ne 0$ for any $x \in (0, 1)$

(d) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$ is always positive $\forall\ x \in (0, 1)$

67. **(d)** $f'(x) = \frac{192x^3}{2 + \sin^4\pi x}$

$\Rightarrow \frac{192x^3}{3} \le f'(x) \le \frac{192x^3}{2} \Rightarrow 64x^3 \le f'(x) \le 96x^3$

$\Rightarrow \int_{1/2}^x 64x^3 dx \le \int_{1/2}^x f'(x)dx \le \int_{1/2}^x 96x^3 dx$

$\Rightarrow \frac{64x^4}{4} - \frac{64}{4}\times\frac{1}{16} \le \int_{1/2}^x f'(x)dx \le \frac{96x^4}{4} - \frac{96}{4\times 16}$

$\Rightarrow 16x^4 - 1 \le \int_{1/2}^x f'(x) \le 24x^4 - \frac{3}{2}$

$\Rightarrow 16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}$

$\Rightarrow \int_{1/2}^1 (16x^4 - 1)dx \le \int_{1/2}^1 f(x)dx \le \int_{1/2}^1\left(24x^4 - \frac{3}{2}\right)dx$

$\Rightarrow \left(\frac{16x^5}{5} - x\right)_{\frac{1}{2}}^1 \le \int_{1/2}^1 f(x)dx \le \left[\frac{24x^5}{5} - \frac{3}{2}x\right]_{\frac{1}{2}}^1$

$\Rightarrow 2.6 \le \int_{1/2}^1 f(x)dx \le 3.9$

$\therefore$ Only (d) is the correct option.

68. **(a, b)** $f(x) = 7\tan^8x + 7\tan^6x - 3\tan^4x - 3\tan^2x$
$= (7\tan^4x - 3)(\tan^4x + \tan^2x)$
$= (7\tan^6x - 3\tan^2x)\sec^2x$

$$\int_0^{\pi/4} f(x)dx = \left[\tan^7 x - \tan^3 x\right]_0^{\pi/4} = 1 - 1 = 0$$

$$\therefore \int_0^{\pi/4} xf(x)dx = \left[x\left(\tan^7 x - \tan^3 x\right)\right]_0^{\pi/4} - \int_0^{\pi/4}\left(\tan^7 x - \tan^3 x\right)dx$$

$$= \int_0^{\pi/4} \tan^3 x(1-\tan^2 x)\sec^2 x\,dx$$

$$= \left[\frac{\tan^4 x}{4} - \frac{\tan^6 x}{6}\right]_0^{\pi/4} = \frac{1}{12}$$

69. **(a, c)** Let $f(t) = e^t(\sin^6 at + \cos^4 at)$

$$\therefore f(k\pi + t) = e^{k\pi+t}(\sin^6 a(k\pi+t) + \cos^4 a(k\pi+t)) = e^{k\pi} f(t)$$

$$\therefore \frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt}$$

$$= \frac{(1+e^{\pi}+e^{2x}+e^{3\pi})\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt}$$

$$= 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi}-1}{e^{\pi}-1}$$

70. **(a, c, d)** $f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$

$$\therefore f'(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{x}{x^2}e^{-\left(\frac{1}{x}+x\right)} = \frac{2}{x}e^{-\left(x+\frac{1}{x}\right)}$$

For $x \in [1, \infty)$, $f'(x) > 0$
$\therefore f$ is monotonically increasing on $[1, \infty)$ (a) is correct.
For $x \in (0, 1)$, $f'(x) > 0$
Hence, f is monotonically increasing on $(0, 1)$.
$\therefore$ (b) is not correct

$$f(x) + f\left(\frac{1}{x}\right) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t} + \int_{x}^{1/x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t} = 0$$

$\therefore$ (c) is correct.

Replacing x by 2^x in $f(x) + f\left(\frac{1}{x}\right) = 0$

We get $f(2^x) + f(2^{-x}) = 0$ or $f(2^x) = -f(2^{-x})$
Hence, $f(2^x)$ is an odd function.
$\therefore$ (d) is correct.

71. **(a, c)** Clearly $g(x)$ may or may not be continuous at $x = a$ or $x = b$.
But it is continuous at all value of x except $x = a, b$.
Let us check the continuity of $g(x)$ at $x = a$ and $x = b$

$$\lim_{x\to a^-} g(x) = 0$$

$$\lim_{x\to a^+} g(x) = \lim_{x\to a^+}\int_a^x f(t)dt = \int_a^a f(t)dt = 0$$

and $g(a) = \int_a^a f(t)dt = 0$

$\therefore g(x)$ is continuous at $x = a$

Also $\lim_{x\to b^-} g(x) = \lim_{x\to b^-}\int_a^x f(t)dt = \int_a^b f(t)dt$

and $\lim_{x\to b^+} g(x) = \int_a^b f(t)dt = \lim_{x\to b^-} g(x) = g(b)$

$\therefore$ $g(x)$ is continuous at $x = b$

Therefore, $g(x)$ is continuous $\forall x \in R$

$$\text{Now } g'(x) = \begin{cases} 0, & x < a \\ f(x), & a \le x \le b \\ 0, & x > b \end{cases}$$

$$g'(a^-) = 0 \text{ and } g'(a^+) = f(a)$$

$$g'(b^-) = f(b) \text{ and } g'(b^+) = 0$$

Since, $f(a), f(b) \in [1, \infty)$ $\therefore f(a), f(b) \neq 0$

$$\therefore g'(a^-) \neq g'(a^+) \text{ and } g'(b^-) \neq g'(b^+)$$

$\Rightarrow$ g is not differentiable at a and b.

72. **(b, c)** We have

$$f(x) = ln x + \int_0^x \sqrt{1+\sin t}\,dt$$

$\Rightarrow f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$ which exists $\forall x \in (0, \infty)$

and $f'(x)$ has finite value $\forall x \in (0, \infty)$, *so* $f'(x)$ is continuous

Also $f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$

Which does not exist at the points where

$\sin x = -1$ like $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

$\therefore f'(x)$ is not differentiable.
$\therefore$ (a) is false but (b) is true

Now $\sqrt{1+\sin t} \ge 0 \Rightarrow \int_0^x \sqrt{1+\sin t}\,dt \ge 0, \forall x \in (0, \infty)$

And $ln x > 0, \forall x \in (1, \infty) \Rightarrow f(x) > 0, \forall x \in (1, \infty)$

For $x \ge e^3$

$$f(x) = ln\, x + \int_0^x \sqrt{1+\sin t}\,dt \ge 3$$

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x} \le \frac{1}{x} + \sqrt{2}, \forall x > 0$$

Now for $x \ge e^3$

$$\Rightarrow 0 < f'(x) \le \frac{1}{x} + \sqrt{2} < \frac{1}{e^3} + \sqrt{2} < 3 \ \forall x \in (e^3, \infty)$$

$\Rightarrow |f'(x)| < |f(x)| \quad \therefore$ (c) is true.

Also $\lim\limits_{x \to \infty} f(x) = \infty$

$\therefore |f(x)| + |f'(x)|$ is not bounded.

$\therefore$ (d) is wrong.

73. **(a)** $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right) dx$

$$= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\tan^{-1} x \right]_0^1 = \frac{22}{7} - \pi$$

74. **(a, b, c)** We have

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x} dx \quad \text{....(i)}$$

$$\Rightarrow I_n = \int_{-\pi}^{\pi} \frac{\sin n(-x)}{(1+\pi^{-x})\sin(-x)} dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \right]$$

$$\Rightarrow I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x} dx \quad \text{...(ii)}$$

Adding equations (i) and (ii), we get

$$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx = 2\int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

[since integrand is an even function]

$$\Rightarrow I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

Now $\displaystyle I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx$

$$= 2\int_0^{\pi} \cos(n+1)x\,dx = 2\left[\frac{\sin(n+1)x}{n+1} \right]_0^{\pi} = 0$$

$\therefore \quad I_{n+2} = I_n$

Also $\displaystyle I_1 = \int_0^{\pi} 1\,dx = \pi$ and $I_0 = 0$

Hence $\displaystyle\sum_{m=1}^{10} I_{2m+1} = I_3 + I_5 + I_7 + ... + I_{21}$

$= 10\, I\ (\because\ I_{n+2} = I_n)$

$= 10\ \pi$

and $\displaystyle\sum_{m=1}^{10} I_{2m} = I_2 + I_4 + I_6 + ... + I_{20}$

$= 20 \times I_0\ \ (\because\ I_{n+2} = I_n)$

$= 20 \times 0 = 0$

75. **(a, b)** Given $g(x) = \displaystyle\int_0^x f(t)\,dt,\ x \in [0, 3]$

$$\Rightarrow g'(x) = f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$$

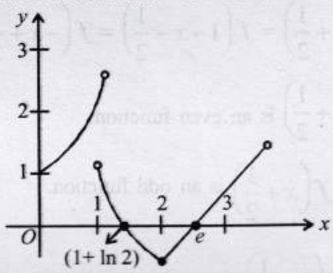
$\therefore \ g'(x) = 0 \Rightarrow e^{x-1} = 2$ or $x - e = 0$

$\Rightarrow x - 1) = \ln 2$ or $x = e \Rightarrow x = 1 + \ln 2$ or e

$$g''(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ -e^{x-1}, & 1 < x \le 2 \\ 1, & 2 < x \le 3 \end{cases}$$

$\therefore \quad g''(1 + \ln 2) = -2$ and $g''(e) = 1$

$\Rightarrow$ $g(x)$ has local max. at $x = 1 + \ln 2$ and local min. at $x = e$.



Graph of $g'(x)$

Also from graph of $g'(x)$, it is clear that $g(x)$ has local max. at $x = 1$ and local min. at $x = 2$

76. **(b, d)** $f(x) = \displaystyle\int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$

$$\frac{dy}{dx} = f'(x) \Rightarrow x(e^x - 1)(x-1)(x-2)^3(x-3)^5 = 0$$

Critical points are 0, 1, 2, 3.

Consider change of sign of $\frac{dy}{dx}$ at $x = 3$.

When $x < 3$, then $\frac{dy}{dx} = -$ ve and when $x > 3$, $\frac{dy}{dx} = +$ve

Change of sign of $\frac{dy}{dx}$ is from $-$ve to $+$ve, hence minimum at $x = 3$. Again minimum and maximum occur alternately.

Hence, 2nd minimum occur at $x = 1$

77. **(a, b, c, d)**

$\therefore f(x)$ is a non constant twice differentiable function such that $f(x) = f(1-x) \Rightarrow f'(x) = -f'(1-x)$ ...(i)

For $x=\frac{1}{2}$, we get $f'\left(\frac{1}{2}\right)=-f'\left(1-\frac{1}{2}\right)$

$\Rightarrow f'\left(\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)=0$ $[\because f(x)=f(1-x)]$

$\Rightarrow f'\left(\frac{1}{2}\right)=0$

$\Rightarrow$ (b) is correct

For $x=\frac{1}{4}$, we get $f'\left(\frac{1}{4}\right)=-f'\left(\frac{3}{4}\right)$ $[\because f(x)=f(1-x)]$

but given that $f'\left(\frac{1}{4}\right)=0$

$\therefore f'\left(\frac{3}{4}\right)=f'\left(\frac{1}{4}\right)=0$

Hence, $f'(x)$ satisfies all conditions of Rolle's theorem for $x\in\left[\frac{1}{4},\frac{1}{2}\right]$ and $\left[\frac{1}{2},\frac{3}{4}\right]$. So there exists at least one point $a_1\in\left(\frac{1}{4},\frac{1}{2}\right)$ and at least one point $b_2\in\left(\frac{1}{2},\frac{3}{4}\right)$. Such that f "$(a_1)=0$ and $f''(b_2)=0$

$\therefore f''(x)$ varishes at least twice on [0, 1] $\Rightarrow$ (a) is correct.

Also using $f(x)=f(1-x)$

$\Rightarrow f\left(x+\frac{1}{2}\right)=f\left(1-x-\frac{1}{2}\right)=f\left(-x+\frac{1}{2}\right)$

$\Rightarrow f\left(x+\frac{1}{2}\right)$ is an even function.

$\Rightarrow \sin x.\ f\left(x+\frac{1}{2}\right)$ is an odd function.

$\Rightarrow \int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\,dx=0$, $\therefore$ (c) is correct.

78. **(a)** $\int_{-1}^{1} f(x)dx=\int_{-1}^{1} x-[x]dx=\int_{-1}^{1} xdx-\int_{-1}^{1}[x]dx$

$=0-\int_{-1}^{1}[x]dx$ ...(i)

$[\because x$ is an odd function$]$

Now $[x]=\begin{cases}-1, & \text{if } -1\le x<0\\ 0, & \text{if } 0\le x<1\\ 1, & \text{if } x=1\end{cases}$

$\therefore \int_{-1}^{1}[x]dx=\int_{-1}^{0}(-1)dx+\int_{0}^{1}0dx=[-x]_{-1}^{0}+0=-1$

Hence from (i) $\int_{-1}^{1} f(x)dx=1$

79. **(a)** $\int_0^x f(t)dt=x+\int_x^1 t\,f(t)dt$

Differentiating both sides w.r.t. x,

$f(x).1-f(0).0=1+1.f(1).0-xf(x).1$

$\Rightarrow (x+1)f(x)=1,\ \Rightarrow f(x)=\frac{1}{x+1}$

$\therefore f(1)=\frac{1}{2}$

80. **(d)** $P(2)$ Let $f(x)=ax^2+bx+c$

where $a, b, c\ge 0$ and a, b, c are integers.

$\because f(0)=0\Rightarrow c=0$

$\therefore f(x)=ax^2+bx$

Also $\int_0^1 f(x)dx=1$

$\Rightarrow \left[\frac{ax^3}{3}+\frac{bx^2}{2}\right]_0^1=1\Rightarrow \frac{a}{3}+\frac{b}{2}=1\Rightarrow 2a+3b=6$

$\because$ a and b are integers

$a=0$ and $b=2$

or $a=3$ and $b=0$

$\therefore$ There are only 2 solutions.

$Q(3)$ $f(x)=\sin x^2+\cos x^2$

$f(x)$ is max. $\sqrt{2}$ at $x^2=\frac{\pi}{4}$ or $\frac{9\pi}{4}$

$\Rightarrow x=\pm\frac{\sqrt{\pi}}{2}$ or $\pm\frac{3\sqrt{\pi}}{2}\in\left[-\sqrt{13},\sqrt{13}\right]$

$\therefore$ There are four points.

$R(1)$ $I=\int_{-2}^{2}\frac{3x^2}{1+e^x}dx=\int_{-2}^{2}\frac{3x^2}{1+e^{-x}}dx$

$\left[\text{Using}\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

$=\int_{-2}^{2}\frac{3x^2e^x}{1+e^x}dx$; $2I=\int_{-2}^{2}\frac{3x^2\left(1+e^x\right)}{1+e^x}dx=\int_{-2}^{2}3x^2dx$

$2I=\left(x^3\right)_{-2}^{2}=8-(-8)=16\Rightarrow I=8$

$S(4)$ $\dfrac{\int_{-1/2}^{1/2}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx}{\int_0^{1/2}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx}=0$

$\because$ Numerator $=0$, function being odd.

Hence option (d) is correct sequence.

81. **(0)** $f(x)=\sin^2 x, g(x)=\sqrt{\frac{\pi}{2}x-x^2}$

Here $f\left(\frac{\pi}{2}-x\right)=\cos^2 x, g\left(\frac{\pi}{2}-x\right)=g(x)$

Let $I_1=2\int_0^{\frac{\pi}{2}} f(x)g(x)=2\int_0^{\frac{\pi}{2}} \sin^2 x.g(x)dx$ ...(i)

as $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$\Rightarrow I_1=2\int_0^{\frac{\pi}{2}}\cos^2 xg(x)dx$ ...(ii)

Euations (i) + (ii)

$\Rightarrow 2I_1=2\int_0^{\frac{\pi}{2}} g(x)dx \quad \Rightarrow I_1=\int_0^{\frac{\pi}{2}} g(x)dx$

$\Rightarrow 2\int_0^{\frac{\pi}{2}} f(x)g(x)-\int_0^{\frac{\pi}{2}} g(x)dx=0$

82. **(0.25)** According to question 16

$2\int_0^{\frac{\pi}{2}} f(x)g(x)dx=\int_0^{\frac{\pi}{2}} g(x)dx=I_1$ (let)

Now, $I_1=\int_0^{\frac{\pi}{2}} g(x)dx=\int_0^{\frac{\pi}{2}}\sqrt{\frac{\pi}{2}x-x^2}\,dx$

$I_1=\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{\pi}{4}\right)^2-\left(\frac{\pi}{4}-x\right)^2}\,dx$

Put $\frac{\pi}{4}-x=t \Rightarrow dx=-dt$

$I_1=-\int_{\frac{-\pi}{4}}^{\frac{-\pi}{4}}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2dt}=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2dt}$

$=2\int_0^{\frac{\pi}{4}}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2dt}$

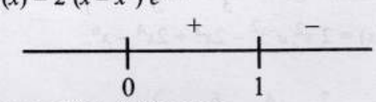
$=2\left[\frac{t}{2}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2}+\frac{\pi^2}{32}\sin^{-1}\left(\frac{4t}{\pi}\right)\right]_0^{\frac{\pi}{4}}=\frac{\pi^3}{32}$

Let $I=\frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} f(x)g(x)dx=\frac{8}{\pi^3}\times I_1=\frac{1}{4}=0.25$

83. **(c)** $\because \quad f(x)=\int_{-x}^{x}(|t|-t^2)e^{-t^2}dt,\ x>0$

$\therefore \quad f(x)=2\int_0^x (t-t^2)e^{-t^2}dt$

$f'(x)=2(x-x^2)e^{-x^2}$ ... (i)



Hence option (d) is wrong.

$g'(x)=xe^{-x^2}2x$ ... (ii)

Add (i) and (ii),

$f'(x)+g'(x)=2xe^{-x^2}$

$f(x)+g(x)=-e^{-x^2}+c$

$\because\ f(0)=g(0)=0$

$f(x)+g(x)=-e^{-x^2}+1$

$f(\sqrt{\ell n3})+g(\sqrt{\ell n3})=1-\frac{1}{3}=\frac{2}{3}$

So, option (a) is wrong.

$H(x)=\psi_1(x)-1-\alpha x=e^{-x}+-1-\alpha x,$

$x\geq 1$ & $\alpha\in(1,x)$

$H(1)=e^{-1}+1-1-\alpha<0$

$H'(x)=-e^{-x}+1-\alpha<0 \quad \Rightarrow H(x)$ is decreasing

So option (b) is wrong.

(c) $\psi_2(x)=2(\psi_1(\beta)-1)$

Applying L.M.V.T to $\psi_2(x)$ in $[0, x]$

$\psi_2'(\beta)=\frac{\psi_2(x)-\psi_2(0)}{x}$

$2\beta-2+2e^{-\beta}=\frac{\psi_2(x)-0}{x}$

$\Rightarrow \quad \psi_2(x)=2x(\psi_1(\beta-1))$ has one solution

So, option (c) is correct.

84. **(d)**

(a) $\psi_1(x)=e^{-x}+x, x\geq 0$

$\psi_1'(x)=1-e^{-x}>0 \Rightarrow \quad \psi_1(x)$ is increasing

$\psi_1(x)\geq\psi_1(0)\forall x\geq 0 \Rightarrow \psi_1(x)\geq 1$

Option (a) is incorrect.

(b) $\psi_2(x)=x^2-2x+2-2e^{-x} \quad x\geq 0$

$\psi_2'(x)=2x-2+2e^{-x}=2\psi_1(x)-2\geq 0, \quad \forall x\geq 0$

$\Rightarrow \quad \psi_2(x)$ is increasing

$\Rightarrow \quad \psi_2(x)\geq\psi_2(0) \quad \Rightarrow \quad \psi_2(x)\geq 0$

So, option (b) is incorrect.

(c) $f(x)=2\int_0^x (t-t^2)e^{-t^2}dt$ & $x\in\left(0,\frac{1}{2}\right)$

$=\int_0^x 2te^{-t^2}dt-\int_0^x 2t^2e^{-t^2}dt$

$=-e^{-x^2}\Big|_0^x-\int_0^x 2t^2e^{-t^2}dt$

Let $H(x)=f(x)-1+e^{-x^2}+\frac{2}{3}x^3-\frac{2}{5}x^5,\ x\in\left(0,\frac{1}{2}\right)$

$H(0)=0$

$H'(x)=2(x-x^2)e^{-x^2}-2xe^{-x^2}+2x^2-2x^4$

$=-2x^2e^{-x^2}+2x^2-2x^4$

$=2x^2(1-x^2-e^{-x^2})$

$\because\ e^{-x}\ge 1-x\ \forall x\ge 0$

$\Rightarrow H'(x)\le 0 \Rightarrow H(x)$ is decreasing

$\Rightarrow 1-1(x)<0\ \forall x\in\left(0,\frac{1}{2}\right)$

Let $P(x)=g(x)-\frac{2}{3}x^3+\frac{2}{5}x^5-\frac{1}{7}x^7\ x\in\left(0,\frac{1}{2}\right)$

$P'(x)=2x^2e^{-x^2}-2x^2+2x^4-x^6$

$=2x^2\left(1-\frac{x^2}{1}+\frac{x^4}{2}-\frac{x^6}{3}+...\right)-2x^2+2x^4-x^6$

$=-\frac{x^8}{3}+\frac{x^{10}}{12}$.........

$\Rightarrow P'(x)\le 0 \Rightarrow P(x)$ is decreasing

$\Rightarrow P(x)\le 0$

So, option (d) is correct.

85. **(a, b, c)** $f(x)=xF(x)\Rightarrow f'(x)=F(x)+xF'(x)$

$\therefore\ f'(1)=F(1)+F'(1)=F'(1)<0$

$\left(\because F'(x)<0, x\in\left(\frac{1}{2},3\right)\right)$

$f(2)=2F(2)<0$, ($\because\ F'(x)<0\Rightarrow F$ is decreasing on $\left(\frac{1}{2},3\right)$

and $F'(1)=0$, $F(3)=-4$)

$f'(x)=F(x)+xF'(x)$

For the same reason given above and $F'(x)<0$ given.

$F(x)<0\ \forall\ x\in(1,3)$

$\therefore\ f'(x)\ne 0, x\in(1,3)$

86. **(c, d)** $\int_1^3 x^2F'(x)dx=-12$

$\Rightarrow \left[x^2F(x)\right]_1^3-\int_1^3 2x\,F(x)\,dx=-12$

$\Rightarrow 9F(3)-F(1)-2\int_1^3 xF(x)dx=-12$

$\Rightarrow \int_1^3 xF(x)dx=-12\Rightarrow\int_1^3 f(x)dx=-12$ ...(i)

Also $\int_1^3 x^3F''(x)dx=40$

$\Rightarrow \left[x^3F'(x)\right]_1^3-3\int_1^3 x^2F'(x)dx=40$

$\Rightarrow \left[x^2(f'(x)-F(x))\right]_1^3-3\times(-12)=40$

$\left\{\begin{array}{l}\text{Using } xF'(x)=f'(x)-F(x)\\ \text{and}\int_1^3 x^2F'(x)dx=-12\end{array}\right.$

$\Rightarrow 9(f'(3)-F(3))-(f'(1)-F(1))=4$

$\Rightarrow 9f'(3)-9\times(-4)-f'(1)+0=4 \Rightarrow 9f'(3)-f'(1)+32=0$

87. **(c)** For the statement P, $f(x)+2x=2(1+x^2)$

$\Rightarrow (1-x)^2\sin^2x+x^2+2x=2(1+x^2)$

$\Rightarrow (1-x)^2\sin^2x=x^2-2x+1+1$

$\Rightarrow (1-x)^2\sin^2x=(1-x)^2+1$

$\Rightarrow (1-x)^2\cos^2x=-1$, which is not possible for any real value of x.

Hence P is not true.

Let $H(x)=2f(x)+1-2x(1+x)$

$H(0)=2f(0)+1-0=1$

and $H(1)=2f(1)+1-4=-3$

Hence, $H(x)$ has a solution in (0, 1)

Therefore, Q is true.

88. **(b)** $g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)dt,\ x\in(1,\infty)$

$\therefore\ g'(x)=\left[\frac{2(x-1)}{x+1}-\ln x\right]f(x)$

Here $f(x)>0$, $\forall\ x\in(1,\infty)$

Let $h(x)=\frac{2(x-1)}{x+1}-\ln x$

$\therefore\ h'(x)=\frac{4}{(x+1)^2}-\frac{1}{x}=\frac{-(x-1)^2}{(x+1)^2x}<0, x\in(1,\infty)$

$\Rightarrow h(x)$ is decreasing function.

Hence, for $x>1$, $h(x)<h(1)\Rightarrow h(x)<0\ \forall\ x>1$

$\Rightarrow g'(x)<0\ \forall\ x\in(1,\infty)$

Therefore, $g(x)$ is decreasing on $(1,\infty)$.

89. **(a)** We have $f(x)=\frac{x^2-ax+1}{x^2+ax+1}; 0<a<2$

$\Rightarrow f'(x)=\frac{2a(x^2-1)}{(x^2+ax+1)^2}$

$\Rightarrow (x^2+ax+1)^2f'(x)=2a(x^2-1)$

$\Rightarrow (x^2+ax+1)^2f''(x)+2(x^2+ax+1)$

$(2x+a)f'(x)=4ax$ ...(i)

Putting $x=-1$ in equation (i), we get

$(2-a)^2f''(-1)=-4a$ ...(ii)

Putting $x=1$ in equation (1), we get

$(2+a)^2f''(1)=4a$ ...(iii)

Adding equations (ii) and (iii), we get

$(2+a)^2f''(1)+(2-a)^2f''(-1)=0$

90. **(a)** We have $f'(x)=\frac{2a(x^2-1)}{(x^2+ax+1)^2}$

$f'(x)=0\Rightarrow x=-1, 1$ are the critical points.

$\xleftarrow{\quad + \quad -1 \quad - \quad 1 \quad + \quad}$

$\therefore f(x)$ is decreasing on $(-1, 1)$

Also using equation (1), $f''(-1) = \dfrac{-4a}{(2-a)^2} < 0$

and $f''(1) = \dfrac{4a}{(2+a)^2} > 0$

$\therefore x = -1$ is a point of local maximum and $x = 1$ is a point of local minimum.

91. **(b)** $g(x) = \displaystyle\int_0^{e^x} \frac{f'(t)}{1+t^2}dt \Rightarrow g'(x) = \frac{f'(e^x)}{1+e^{2x}} \cdot e^x$

$$= \frac{2a(e^{2x}-1)e^x}{(e^{2x}+ae^x+1)^2(1+e^{2x})} = \frac{2ae^x}{1+e^{2x}} \cdot \frac{e^{2x}-1}{(e^{2x}+ae^x+1)^2}$$

Now $g'(x) > 0$ for $e^{2x} - 1 > 0 \Rightarrow x > 0$

and $g'(x) < 0$ for $e^{2x} - 1 < 0 \Rightarrow x < 0$

$\therefore g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

92. **(a)** $\displaystyle\int_0^{\pi/2} \sin x\,dx = \frac{\left(\frac{\pi}{2}-0\right)}{4}\left(\sin 0 + \sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}\right)$

$$= \frac{\pi}{8}(1+\sqrt{2})$$

93. **(d)** $\displaystyle\lim_{x\to a} \frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$

$$\lim_{h\to 0} \frac{\int_a^{a+h} f(x)dx - \frac{h}{2}(f(a+h)+f(a))}{h^3} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h) - \frac{1}{2}[f(a)+f(a+h)] - \frac{h}{2}(f'(a+h))}{3h^2} = 0$$

[Using L'Hopital rule]

$$\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h} = 0$$

[Using L'Hopital rule]

$$\Rightarrow \lim_{h\to 0} -\frac{f''(a+h)}{12} = 0 \Rightarrow f''(x) = 0, \forall a \in R$$

$\Rightarrow f(x)$ must be of max. degree 1.

94. **(b)** $f''(x) < 0, \forall x \in (a, b)$, for $c \in (a, b)$

$$F(c) = \frac{c-a}{2}(f(a)+f(c)) + \frac{b-c}{2}(f(b)+f(c))$$

$$= \frac{b-a}{2}f(c) + \frac{c-a}{2}f(a) + \frac{b-c}{2}f(b)$$

$$\Rightarrow F'(c) = \frac{b-a}{2}f'(c) + \frac{1}{2}f(a) - \frac{1}{2}f(b)$$

$$= \frac{1}{2}[(b-a)f'(a) - f(b)]$$

$$F''(c) = \frac{1}{2}(b-a) \Rightarrow f''(c) < 0$$

$[\because f''(x) < 0, \forall x \in (a,b)$ and $b > a]$

$\therefore$ $F(c)$ is max. at the point $(c, f(c))$ where $F'(c) = 0$

$$\Rightarrow f'(c) = \frac{1}{2}\left(\frac{f(b)-f(a)}{b-a}\right).$$

95. Let $I = \displaystyle\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}dx$

$$= \int_{-\pi/3}^{\pi/3} \frac{\pi}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx$$

The second integral becomes zero integrand being an odd function of x.

$$= 2\pi\int_0^{\pi/3} \frac{dx}{2-\cos\left(x+\frac{\pi}{3}\right)}$$

{using the prop. of even function and also $|x| = x$ for $0 \le x \le \pi/3$}

Let $x + \pi/3 = y \Rightarrow dx = dy$

also as $x \to 0$, $y \to \pi/3$ as $x \to \pi/3$, $y \to 2\pi/3$

$\therefore$ The given integral becomes

$$= 2\pi\int_{\pi/3}^{2\pi/3} \frac{dy}{2-\cos y} = 2\pi\int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \frac{1-\tan^2 y/2}{1+\tan^2 y/2}}$$

$$= 2\pi\int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{3\tan^2 y/2 + 1}dy$$

$$= \frac{2\pi}{3}\int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{\tan^2 y/2 + (1/\sqrt{3})^2}dy$$

$$= \frac{4\pi\sqrt{3}}{3}\left[\tan^{-1}(\sqrt{3}\tan y/2)\right]_{\pi/3}^{2\pi/3} = \frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3 - \pi/4\right]$$

96. We have,

$$y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

Since, $\cos x$ is independent of θ

$$\therefore y(x) = \cos x\int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

$$\Rightarrow \frac{dy}{dx} = -\sin x \int_{\pi^2/16}^{x^2} \left[\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\right] d\theta$$

$$+\cos x \frac{d}{dx}\left[\int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta\right]$$

$$= -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$$

$$+\cos x\left[\frac{\cos x}{1+\sin^2 x}.2x - 0\right] \text{ (By Leibnitz theorem)}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\pi} = -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} . d\theta + \frac{(\cos^2\pi).2\pi}{1+\sin^2\pi}$$

$= 2\pi$

97. Let $I = \int_0^{\pi/2} f(\cos 2x)\cos x\, dx$ ...(i)

$$\Rightarrow I = \int_0^{\pi/2} f\left(\cos 2\left(\frac{\pi}{2}-x\right)\right)\cdot\cos\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow I = \int_0^{\pi/2} f(\cos 2x)\sin x\, dx \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\cos 2x)(\sin x + \cos x)dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\cos 2x)\left[\cos(x-\pi/4)\right]dx$$

Put $-x+\frac{\pi}{4} = t \Rightarrow -dx = dt$

$$\therefore \quad 2I = -\sqrt{2}\int_{\pi/4}^{-\pi/4} f\left[\cos\left(\frac{\pi}{2}-2t\right)\right]\cos t\, dt$$

$$\Rightarrow \quad 2I = \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\sin 2t)\cos t\, dt$$

$$\because \quad \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

$$\because \quad I = \sqrt{2}\int_0^{\pi/4} f(\sin 2t)\cos t\, dt$$

98. Given : $F(x) = \int_0^x f(t)dt$

$\therefore \quad F'(x) = f(x).1 - f(0).0$

[using Leibnitz theorem]

$\Rightarrow \quad F'(x) = f(x)$ ... (i), $\forall x \ge 0$

Also $F(0) = \int_0^0 f(t)dt = 0$

Now $f(x) \le cF(x), \forall x \ge 0$ (Given)

$\therefore \quad f(0) \le cF(0) = 0, \quad \therefore \quad f(0) \le 0$ ....(ii)

Now given that $f(x)$ is non- negative continuous function on $[0, \infty)$

$\therefore \quad f(x) \ge 0, \quad \therefore \quad f(0) \ge 0$ ....(iii)

$\therefore$ From (ii) and (iii) $f(0) = 0$

Also given that $f(x) \le cF(x) \forall x \ge 0$

$\Rightarrow \quad f(x) - cF(x) \le 0$

$\Rightarrow \quad F'(x) - cF(x) \le 0, \forall x \ge 0$ [using (i)]

$e^{-cx}F'(x) - ce^{-cx}F(x) \le 0$

[By multiplying both sides by e^{-cx} (I.F.) and keeping in mind that $e^{-cx} > 0, \forall x$]

$\Rightarrow \quad \frac{d}{dx}[e^{-cx}F(x)] \le 0$

$\Rightarrow \quad g(x) = e^{-cx}F(x)$ is a decreasing function on $[0, \infty)$ i.e., $g(x) \le g(0)$ for all $x \ge 0$

But $g(0) = F(0) = 0, \quad \therefore \quad g(x) \le 0, \forall x \ge 0$

$\Rightarrow \quad e^{-cx}F(x) \le 0, \forall x \ge 0 \Rightarrow F(x) \le 0, \forall x \ge 0$

$\therefore \quad f(x) \le cF(x) \le 0, \forall x \ge 0$

$\Rightarrow \quad f(x) \le 0, \forall x \ge 0$

But given that $f(x) \ge 0 \Rightarrow f(x) = 0, \forall x \ge 0.$

99. $f(x) = \int_1^x \frac{\ln t}{1+t}dt$ for $x > 0$ (given)

Now $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1+t}dt$: Put $t = \frac{1}{u}$, so that

$dt = -\frac{1}{u^2}du$

Therefore $f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln(1/u)}{1+\frac{1}{u}}.\frac{(-1)}{u^2}du$

$$= \int_1^x \frac{\ln u}{u(u+1)}du = \int_1^x \frac{\ln t}{t(t+1)}dt$$

Now, $f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln t}{1+t}dt + \int_1^x \frac{\ln t}{t(1+t)}dt$

$$= \int_1^x \frac{(1+t)\ln t}{t(1+t)}dt = \frac{1}{2}(\ln t)^2]_1^x = \frac{1}{2}(\ln x)^2$$

Put $x = e$, hence $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}(\ln e)^2 = \frac{1}{2}$

Hence Proved.

100. $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}dx$ ...(i)

$$\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$I = \int_0^{\pi} \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi-x)}} \Rightarrow I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} \quad ...(ii)$$

Adding (i) and (ii), $2I = \int_0^{\pi} dx = \pi \Rightarrow I = \pi/2$

101. $\therefore I=\int_0^1 y\,dx=\int_0^1 \tan^{-1}x\,dx-\int_0^1 \tan^{-1}(x-1)dx$

$=\int_0^1 \tan^{-1}x\,dx-\int_0^1 \tan^{-1}\{(1-x)-1\}$

Since, $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$=\int_0^1 \tan^{-1}x\,dx-\int_0^1(-\tan^{-1}x)dx=2\int_0^1 \tan^{-1}x\,dx$ (Proved)

$=2\left[x\tan^{-1}x\,dx-\frac{1}{2}\log(1+x^2)\right]_0^1$

$=\frac{\pi}{2}-\log 2$ ...... (i)

Now, $\int_0^1 \tan^{-1}(1-x+x^2)dx$

$=\int_0^1 \cot^{-1}\frac{1}{1-x+x^2}dx=\int_0^1\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{1-x+x^2}\right)dx$

$=\left[\frac{\pi}{2}x\right]_0^1-I=\frac{\pi}{2}-\left(\frac{\pi}{2}-\log 2\right)=\log 2$ by (i)

102. Let $b-a=t$, where $a+b=4$

$\Rightarrow\ a=\frac{4-t}{2}$ and $b=\frac{t+4}{2}$

Since $a<2$ and $b>2\Rightarrow t>0$

Now $\int_0^a g(x)\,dx+\int_0^b g(x)\,dx$

$=\int_0^{\frac{4-t}{2}} g(x)\,dx+\int_0^{\frac{4+t}{2}} g(x)\,dx=\phi(t)$ [say]

$\Rightarrow\ \phi'(t)=g\left(\frac{4-t}{2}\right)\left(-\frac{1}{2}\right)+g\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right)$

$\left[\because \frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)dt\right]=f[v(x)]\cdot v'(x)-f[u(x)]\cdot u'(x)\right]$

$=\frac{1}{2}\left[g\left(\frac{4+t}{2}\right)-g\left(\frac{4-t}{2}\right)\right]$

Since $g(x)$ is an increasing function (given)

$\therefore$ for $x_1>x_2\Rightarrow g(x_1)>g(x_2)$

Here we have $\left(\frac{4+t}{2}\right)>\left(\frac{4-t}{2}\right)$

$\Rightarrow\ g\left(\frac{4+t}{2}\right)>g\left(\frac{4-t}{2}\right)$

$\Rightarrow\ \phi'(t)=\frac{1}{2}\left[g\frac{(4+t)}{2}-g\frac{(4-t)}{2}\right]>0\Rightarrow \phi'(t)>0$

Hence $\phi(t)$ increase as t increases.

$\Rightarrow\ \int_0^a g(x)\,dx+\int_0^b g(x)\,dx$ increases as $(b-a)$ increases.

103. $\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx=I$ (say)

or $I=\int_{-\pi}^{\pi}\frac{2x}{1+\cos^2 x}dx+\int_{-\pi}^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$

$I=0+2\int_0^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx\quad\left[\because \frac{2x}{1+\cos^2 x}\text{ is an odd function}\right]$

or $I=4\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$ .....(i)

or $I=4\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\{\cos(\pi-x)\}^2}dx$

$=4\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

or $I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx-I$ [from (i)]

or $2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

Putting $\cos x=t$, $-\sin x\,dx=dt$

When $x\to 0, t\to 1$ and when $x\to\pi, t\to -1$

$\Rightarrow I=2\pi\int_1^{-1}\frac{-dt}{1+t^2}=2\pi\int_{-1}^{1}\frac{dt}{1+t^2}$

$\because \int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx$

$I=4\pi\int_0^1\frac{dt}{1+t^2}$

$\Rightarrow I=4\pi\left(\tan^{-1}t\right)_0^1=4\pi\{\tan^{-1}(1)-\tan^{-1}(0)\}$

$\Rightarrow I=4\pi\left\{\frac{\pi}{4}-0\right\}=\pi^2$

104. Let $I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

We know that $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$

Also $\sin^{-1}y+\cos^{-1}y=\frac{\pi}{2}$

$\therefore$ We get $\frac{\pi}{2}-\cos^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$

$\Rightarrow \cos^{-1}\left(\frac{2x}{1+x^2}\right)=\frac{\pi}{2}-2\tan^{-1}x$

$\therefore I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\left[\frac{\pi}{2}-2\tan^{-1}x\right]dx$

$=\frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx-2\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4\tan^{-1}x}{1-x^4}dx$

$$= 2.\frac{\pi}{2}\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx - 2\times 0$$

$$\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f \text{ is even}$$
$$= 0 \text{ if } f \text{ is odd}$$

$$\therefore I = \pi\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx$$

$$\therefore I = -\pi\int_0^{1/\sqrt{3}} \frac{(1-x^4)-1}{1-x^4}dx$$

$$= -\pi\int_0^{1/\sqrt{3}} 1-\frac{1}{1-x^4}dx = -\pi\int_0^{1/\sqrt{3}}\left[1-\frac{1}{2}\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)\right]dx$$

$$= -\pi\left[x-\frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1+x}{1+x}\right|+\tan^{-1}x\right)\right]_0^{1/\sqrt{3}}$$

$$= -\pi\left[\frac{1}{\sqrt{3}}-\frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}\right|-\tan^{-1}\frac{1}{\sqrt{3}}\right)-0\right]$$

$$= \frac{\pi}{12}[\pi+3\log(2+\sqrt{3})-4\sqrt{3}]$$

105. $I = \int_2^3 \frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx$

$$= \int_2^3 \frac{2x^3(x^2-1)+(x^2+1)^2}{(x^2+1)^2(x^2-1)}dx$$

$$= \int_2^3 \frac{2x^3}{(x^2+1)^2} + \int_2^3 \frac{1}{x^2-1}dx$$

$$= \int_2^3 \frac{x^2.2x}{(x^2+1)^2} + \left[\frac{1}{2}\log\frac{x-1}{x+1}\right]_2^3$$

Put $x^2+1=t,\ 2x\,dx = dt$

when $x \to 2, t\to 5,\ x\to 3, t\to 10$

$$\therefore I = \int_5^{10}\frac{t-1}{t^2}dt + \frac{1}{2}\left(\log\frac{2}{4}-\log\frac{1}{3}\right)$$

$$= \int_5^{10}\left(\frac{1}{t}-\frac{1}{t^2}\right)dt + \frac{1}{2}\log\frac{3}{2} = \left(\log|t|+\frac{1}{t}\right)_5^{10} + \frac{1}{2}\log\frac{3}{2}$$

$$= \log 10 - \log 5 + \frac{1}{10} - \frac{1}{5} + \frac{1}{2}\log\frac{3}{2}$$

$$= \frac{1}{2}\left[2\log 2 + \log\frac{3}{2}\right] - \frac{1}{10}$$

$$= \frac{1}{2}\log 6 - \frac{1}{10}$$

106. It is given that $\int_0^1 e^x(x-1)^n\,dx = 16-6e$

where $n \in N$ and $n \le 5$

To find the value of n.

Let $I_n = \int_0^1 e^x(x-1)^n\,dx$

$$= [(x-1)^n e^x]_0^1 - \int_0^1 n(x-1)^{n-1}e^x\,dx$$

$$= -(-1)^n - \int_0^1 n(x-1)^{n-1}e^x\,dx$$

$$I_n = (-1)^{n+1} - n\,I_{n-1} \quad \text{...(i)}$$

Also, $I_1 = \int_0^1 e^x(x-1)\,dx$

$$= [e^x(x-1)]_0^1 - \int_0^1 e^x\,dx = -(-1)-(e^x)_0^1$$

$$= -(e-1) = 2-e$$

Using eq. (i), $I_2 = (-1)^3 - 2\,I_1 = -1-2\,(2-e) = 2e-5$

Similarly, $I_3 = (-1)^4 - 3I_2 = 1-3(2e-5) = 16-6e$

$\therefore n = 3$

107. Let $I = \int_0^{\pi} \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

Consider, $2x-\pi = y \Rightarrow dx = \frac{dy}{2}$. Also, $x = \left(\frac{\pi}{2}+\frac{y}{2}\right)$

When $x\to 0, y\to -\pi$ when $x\to\pi, y\to\pi$

$\therefore$ We get

$$I = \int_{-\pi}^{\pi} \frac{\left(\frac{\pi+y}{2}\right)\sin(\pi+y)\sin\left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}+\frac{y}{2}\right)\right]}{y}.\frac{dy}{2}$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\left(\frac{\pi}{y}+1\right)(-\sin y)\sin\left(\frac{-\pi}{2}\sin\frac{y}{2}\right)dy$$

$$= \frac{\pi}{4}\int_{-\pi}^{\pi}\frac{\sin y\sin(\pi/2\sin y/2)}{y}dy$$

$$+\frac{1}{4}\int_{-\pi}^{\pi}\sin y\sin\left(\frac{\pi}{2}\sin\frac{y}{2}\right)dy$$

$$\because \int_{-a}^{a} f(x)\,dx = 0 \text{ if f is odd function}$$

$$= 2\int_0^a f(x)\,dx \text{ if } f \text{ is an even function]}$$

$$\therefore I = 0 + \frac{2}{4}\int_0^{\pi}\sin y\sin(\pi/2\sin y/2)\,dy$$

$$\therefore I = \frac{1}{2}\int_0^{\pi} 2\sin y/2\cos y/2\sin(\pi/2\sin y/2)\,dy$$

Let $\sin y/2 = u \Rightarrow \frac{1}{2}\cos y/2\,dy = du$

$\Rightarrow \cos y/2\,dy = 2du$

Also as $y\to 0, u\to 0$ and as $y\to\pi, u\to 1$

$\therefore I = \int_0^1 2u \sin\left(\frac{\pi u}{2}\right) du$

$$= \left[2u \frac{-\cos\frac{\pi u}{2}}{\pi/2}\right]_0^1 + \int_0^1 2.\frac{2}{\pi}\cos\left(\frac{\pi u}{2}\right) du$$

$$= 0 + \frac{4}{\pi}\left.\frac{\sin\left(\frac{\pi u}{2}\right)}{\pi/2}\right|_0^1 = \frac{8}{\pi^2}$$

108. We are given that f is a continuous function and $\int_0^x f(t)dt \to \infty$ as $|x| \to \infty$

To show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)dt = 2.$ ...(i)

Let $y = mx$ intersects the given curve, then put $y = mx$ in the equation (i) of the curve

$$m^2x^2 + \int_0^x f(t)dt = 2 \quad ...(ii)$$

Let $F(x) = m^2x^2 + \int_0^x f(t)dt - 2$

Then $F(x)$ is a continuous function as $f(x)$ is given to be continuous.

Also $F(x) \to \infty$ as $|x| \to \infty$

But $F(0) = -2$

Thus $F(0) = -$ve and $F(b) = +$ve where b is some value of x, and $F(x)$ is continuous.

Therefore $F(x) = 0$ for some value of $x \in (0, b)$ or eq. (i) is solvable for x.

Hence $y = mx$ intersects the given curve.

109. $\because$ 2 sin x [cos x + cos $3x$ + cos $5x$ ... + cos $(2k-1)x$]

= 2 sin x cos x + 2 sin x cos $3x$ + 2sin x cos $5x$ + ... + 2 sin x cos $(2k-1)x$

= sin $2x$ + (sin $4x$ – sin $2x$) + (sin $6x$ – sin $4x$) + ... + {sin $2kx$ – sin $(2k-2)x$}

= sin $2kx$

$\therefore$ 2 [cos x + cos $3x$ + cos $5x$ + ... + cos $(2k-1)x$]

$$= \frac{\sin 2kx}{\sin x} \quad ...(i)$$

Now, $\sin 2kx.\cot x = \frac{\sin 2kx}{\sin x}.\cos x$

= 2 cos x [cos x + cos $3x$ + cos $5x$ + ... + cos $(2k-1)x$] [from eqn (i)]

= [2 cos² x + 2 cos x cos $3x$ + 2 cos x cos $5x$ + ... + 2 cos x cos $(2k-1)x$]

= (1 + cos2x) + (cos $4x$ + cos $2x$) + (cos $6x$ + cos $4x$) + ... + {cos $2kx$ + cos $(2k-2)x$}

= 1 + 2 [cos $2x$ + cos $4x$ + cos $6x$ + ...+ cos $(2k-2)x$] + cos $2kx$

$$\therefore \int_0^{\pi/2} (\sin 2kx).\cot x\, dx$$

$$= \int_0^{x/2} 1.dx + 2\int_0^{\pi/2} (\cos 2x + \cos 4x ... \cos(2k-2)x)dx + \int_0^{\pi/2} \cos(2k)x\, dx$$

$$= \frac{\pi}{2} + 2\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + ... + \frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2} + \left[\frac{\sin(2k)x}{2k}\right]_0^{\pi/2} = \frac{\pi}{2} + 2\times 0 = \frac{\pi}{2}$$

110. We have, $I = \int_0^{\pi/2} f(\sin 2x)\sin x dx$ ...(i)

$I = \int_0^{\pi/2} f(\sin 2x)\cos x dx$ ...(ii)

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)(\cos x + \sin x)dx$$

$$\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ when } f(2a-x) = f(x)]$$

$$\therefore \quad 2I = 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$$

$$\Rightarrow I = \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$$

$$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin(\pi/4 + x)dx$$

$$\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\therefore \quad I = \sqrt{2}\int_0^{\pi/4} f\left[\sin\left(2\left(\frac{\pi}{4} - x\right)\right)\right]\sin(\pi/4 + \pi/4 - x)dx$$

$$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x dx$$

Hence Proved.

111. Let $I=\int_0^a f(x)g(x)dx=\int_0^a f(a-x)g(a-x)dx$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\therefore\ I=\int_0^a f(x)(2-g(x))dx$$

It is given that $f(a-x)=f(x)$ and $g(a-x)+g(x)=2$

$$\therefore\ I=2\int_0^a f(x)dx-\int_0^a f(x)g(x)dx;$$

$$\Rightarrow I=2\int_0^a f(x)dx-I \Rightarrow I=\int_0^a f(x)dx$$

Hence the result.

112. Let $I=\int_0^1 1.\log[\sqrt{1-x}+\sqrt{1+x}]dx$

Intergrating by parts, we get

$$I=[x\log(\sqrt{1-x}+\sqrt{1+x})]_0^1$$

$$-\int_0^1 x.\frac{1}{\sqrt{1-x}+\sqrt{1+x}}.\left[\frac{-1}{2\sqrt{1-x}}+\frac{1}{2\sqrt{1+x}}\right]dx$$

$$=\log\sqrt{2}-\int_0^1 x\frac{(\sqrt{1+x}-\sqrt{1-x})}{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}-\sqrt{1-x})}.\frac{(\sqrt{1-x}-\sqrt{1+x})}{2\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}\log 2+\frac{1}{2}\int_0^1\frac{1+x+1-x-2\sqrt{1-x^2}}{2\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}\log 2+\frac{1}{2}\int_0^1\frac{1}{\sqrt{1-x^2}}dx-\frac{1}{2}\int_0^1 1dx$$

$$=\frac{1}{2}\left[\log 2+\left(\sin^{-1}x\right)_0^1-(x)_0^1\right]=\frac{1}{2}[\log 2+\pi/2-1]$$

113. $f(x)=\int_1^x[2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]dt$

$$\because\ \frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)}g(t)dt\right]=g[\psi(x)]\psi'(x)-g[\phi(x)]\phi'(x)$$

$\therefore\ f'(x)=2(x-1)(x-2)^3+3(x-1)^2(x-2)^2$
$=(x-1)(x-2)^2(2x-4+3x-3)$
$=(x-1)(x-2)^2(5x-7)$

For extreme values, put $f'(x)=0\Rightarrow x=1,2,7/5$

Now, $f''(x)=(x-2)^2(5x-7)+2(x-1)(x-2)(5x-7)+5(x-1)(x-2)^2$

$\because\ f''(1)=1(-2)=-2<0,\ \therefore f$ is max. at $x=1$

$\because\ f''(2)=0$

$\therefore\ f$ is neither maximum nor minimum at $x=2$.

$$f''\left(\frac{7}{5}\right)=5\left(\frac{7}{5}-1\right)\left(\frac{7}{5}-2\right)^2=5\times\frac{2}{5}\times\frac{9}{25}=\frac{18}{25}>0$$

$\therefore\ f(x)$ is minimum at $x=7/5$.

114. Let $I=\int_0^\pi\frac{x\,dx}{1+\cos\alpha\sin x}$ ...(i)

$$I=\int_0^\pi\frac{(\pi-x)dx}{1+\cos\alpha(\sin(\pi-x))}$$

$$\left[\text{Using }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$\therefore\ I=\int_0^\pi\frac{(\pi-x)dx}{1+\cos\alpha\sin x}$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_0^\pi\frac{x+\pi-x}{1+\cos\alpha\sin x}dx=\int_0^\pi\frac{\pi}{1+\cos\alpha\sin x}dx$$

$$\therefore\ I=\frac{\pi}{2}\int_0^\pi\frac{1}{1+\cos\alpha\sin x}dx$$

$$=\pi\int_0^{\pi/2}\frac{1}{1+\cos\alpha.\dfrac{2\tan x/2}{1+\tan^2 x/2}}dx$$

$$=\pi\int_0^{\pi/2}\frac{\sec^2 x/2}{1+\tan^2 x/2+2\cos\alpha\tan x/2}dx$$

Put $\tan x/2=t$, $\frac{1}{2}\sec^2\frac{x}{2}dx=dt\Rightarrow \sec^2 x/2dx=2dt$

Also when $x\to 0, t\to 0$ as $x\to\pi/2, t\to 1$

$$\therefore\ I=\pi\int_0^1\frac{2dt}{t^2+(2\cos\alpha)t+1}$$

$$=2\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+1-\cos^2\alpha}$$

$$=2\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$$

$$=2\pi.\frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^1$$

$$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right)-\tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

$$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}(\cot\alpha/2)-\tan^{-1}(\cot\alpha)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}(\tan^{-1}(\pi/2-\alpha/2)) - \tan^{-1}(\tan(\pi/2-\alpha))\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right] = \frac{\pi\alpha}{\sin\alpha}$$

115. Let $I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ ... (i)

$$I = \int_0^{\pi/2} \frac{(\pi/2-x)\sin(\pi/2-x)\cos(\pi/2-x)}{\cos^4(\pi/2-x)+\sin^4(\pi/2-x)}dx$$

Since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Then, $I = \int_0^{\pi/2} \frac{(\pi/2-x)\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$\Rightarrow I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^4 x\left(\frac{\sin^4 x}{\cos^4 x}+1\right)}dx = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sec^2 x\tan x}{\tan^4 x+1}dx$$

$$= \frac{\pi}{2\times 4}\int_0^{\pi/2} \frac{2\tan x\sec^2 x\,dx}{1+(\tan^2 x)^2}$$

Put $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\,dx = dt$

Also as $x\to 0, t\to 0$; as $x\to \pi/2, t\to\infty$

$$\therefore\ I = \frac{\pi}{8}\int_0^{\infty}\frac{dt}{1+t^2} = \frac{\pi}{8}[\tan^{-1}t]_0^{\infty} = \frac{\pi}{8}[\pi/2-0] = \pi^2/16$$

116. Let $\int f(x)dx = F(x)+c$

Then $F'(x) = f(x)$ ... (i)

Now $I = \int_a^{a+t} f(x)dx = F(a+t)-F(a)$

$$\therefore\ \frac{dI}{da} = F'(a+t)-F(a) = f(a+t)-f(a)$$

[Using eq. (i)]

$= f(a)-f(a) = 0$

[Using given condition]

This shows that I is independent of a.

117. Let $I = \int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

Also when $x = 0$, $\theta = 0$

and when $x = 1/2, \theta = \pi/6$

Thus, $I = \int_0^{\pi/6}\frac{\sin\theta\sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta$

$$= \int_0^{\pi/6}\frac{\theta\sin\theta}{\cos\theta}\cos\theta d\theta \Rightarrow I = \int_0^{\pi/6}\theta\sin\theta d\theta$$

Intergrating the above by parts, we get

$$I = [\theta(-\cos\theta)]_0^{\pi/6} + \int_0^{\pi/6} 1.\cos\theta d\theta$$

$$= [-\theta\cos\theta+\sin\theta]_0^{\pi/6} = \frac{-\pi}{6}.\frac{\sqrt{3}}{2}+\frac{1}{2} = \frac{6-\pi\sqrt{3}}{12}$$

118. $I = \int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

Let $\sin x - cox x = t \Rightarrow$ as $x\to 0, t\to -1$ as $x\to\pi/4, t\to 0$

$\Rightarrow (\cos x+\sin x)dx = dt$

Also, $t^2 = 1-\sin 2x \Rightarrow \sin 2x = 1-t^2$

$$I = \int_{-1}^{0}\frac{dt}{9+16(1-t^2)} = \frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$$

$$= \frac{1}{40}\left[\log 1-\log\frac{1}{9}\right] = \frac{2\log 3}{40} = \frac{1}{20}\log 3$$

119. $\int_{-1}^{3/2}|x\sin\pi x|dx$

For $-1\le x<0 \Rightarrow -\pi<\pi x<0 \Rightarrow \sin\pi x<0$

$\Rightarrow x\sin\pi x>0$

For $1<x<3/2$

$\Rightarrow \pi<\pi x<3\pi/2 \Rightarrow \sin\pi x<0 \Rightarrow x\sin\pi x<0$

$$\therefore \int_{-1}^{3/2}|x\sin\pi x|dx = \int_{-1}^{1}x\sin\pi x\,dx + \int_{1}^{3/2}(-x\sin\pi x)ax$$

$$= 2\int_0^1 x\sin\pi x\,dx - \int_1^{3/2}x\sin\pi x\,dx$$

$$= 2\left[\frac{-x\cos\pi x}{\pi}+\frac{\sin\pi x}{\pi^2}\right]_0^1 - \left[\frac{-x\cos\pi x}{\pi}+\frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

$$= 2\left[\frac{1}{\pi}\right]-\left[-\frac{1}{\pi^2}-\frac{1}{\pi}\right] = \frac{3}{\pi}+\frac{1}{\pi^2}$$

120. Let $I=\int_0^{\pi} xf(\sin x)dx$ ... (i)

$\Rightarrow I=\int_0^{\pi}(\pi-x)f(\sin x)dx$ ...(ii)

Adding (i) and (ii), we get,

$$2I=\int_0^{\pi}\pi f(\sin x)dx$$

$$I=\frac{\pi}{2}\int_0^{\pi}f(\sin x)dx$$

Hence Proved.

Topic-4: Summation of Series by Integration

1. **(b)** $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\,ln(1+t)}{t^4+4}dt$ $\left[\frac{0}{0}\text{form}\right]$

Using L' Hospital's rule, we get

$$\lim_{x\to 0}\frac{\frac{x ln(1+x)}{x^4+4}}{3x^2}=\lim_{x\to\infty}\frac{ln(1+x)}{x}\cdot\frac{1}{3(x^4+4)}=1\cdot\frac{1}{12}=\frac{1}{12}$$

2. **(a)** We have $I_{m,n}=\int_0^1 t^m(1+t)^n dt$

$$I_{m,n}=\left[\frac{t^{m+1}}{m+1}(1+t)^n\right]_0^1-\frac{n}{m+1}\int_0^1 t^{m+1}(1+t)^{n-1}dt$$

[Using intergrating by parts]

$$I_{m,n}=\frac{2^n}{m+1}-\frac{n}{m+1}I_{m+1,n-1}$$

3. **(5)** $f(t)=\left(\frac{(2n+1)-t}{2}\right)(-1)^{n+1}2+\left(\frac{t-(2n-1)}{2}\right)$
$(-1)^{n+2}2,\ t\in(2n-1,2n+1)$

$\Rightarrow f(t)=2(-1)^{n+1}(2n-t),\ t\in(2n-1,2n+1)$

We have, $g(x)=\int_1^x f(t)dt, x\in(1,8]$

$$=\begin{cases}\int_1^x 2(2-t)dt, 1<x\le 3, n=1\\ \int_1^3 2(2-t)dt+\int_3^x(2t-8)dt, 3<x\le 5, n=2\\ \int_1^3 2(2-t)dt+\int_3^5(2t-8)dt+\int_5^x 2(6-t)dt,\\ \qquad 5<x\le 7, n=3\\ \int_1^3 2(2-t)dt+\int_3^5(2t-8)dt+\int_5^7 2(6-t)dt+\\ \qquad\int_7^x(2t-16)dt, x\in(7,8], n=4\end{cases}$$

$$=\begin{cases}-x^2+4x-3, 1<x\le 3,\\ x^2-8x+15, 3<x\le 5\\ -x^2+12x-35, 5<x\le 7\\ x^2-16x+63, 7<x\le 8\end{cases}$$

$$=\begin{cases}-(x-1)(x-3), 1<x\le 3\\ (x-3)(x-5), 3<x\le 5\\ -(x-5)(x-7), 5<x\le 7\\ (x-7)(x-9), 7<x\le 8\end{cases}$$

$\Rightarrow g(x)=0\Rightarrow x=3,5,7\Rightarrow\alpha=3$

$$\beta=\lim_{x\to 1^+}\left(\frac{g(x)}{x-1}\right)=\lim_{x\to 1^+}-\frac{(x-1)(x-3)}{x-1}=2$$

$\Rightarrow\alpha+\beta=5$

4. **(0)** Given, $f(x)=\int_0^{x\tan^{-1}x}\frac{e^{t-\cos t}}{1+t^{2023}}dt$

$$f'(x)=\frac{e^{x\tan^{-1}x-\cos(x\tan^{-1}x)}}{1+(x\tan^{-1}x)^{2023}}\cdot\left(\frac{x}{1+x^2}+\tan^{-1}x\right)$$

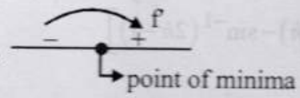
For $x<0$, $\tan^{-1}x\in\left(-\frac{\pi}{2},0\right)$

For $x\ge 0$, $\tan^{-1}x\in\left[0,\frac{\pi}{2}\right]$

$\Rightarrow x\tan^{-1}x\ge 0,\ \forall x\in R$

And $\frac{x}{1+x^2}+\tan^{-1}x=\begin{cases}>0 & \text{For } x>0\\ <0 & \text{For } x<0\\ 0 & \text{Fox } x=0\end{cases}$

i.e



So, f(x) is minimum at x = 0.

Here minimum value is $f(0) = \int_0^0 = 0$.

5. **(57.00)** $f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j\, dt$

$$f_1'(x) = \prod_{j=1}^{21} (x-j)^j = (x-1)(x-2)^2(x-3)^3 \ldots (x-21)^{21}$$

Checking the sign scheme of $f_1'(x)$ at $x = 1, 2, 3, \ldots 21$

We get

$f_1(x)$ has local minima at $x = 1, 5, 9, 13, 17, 21$ and local maxima at 3, 7, 11, 15, 19.

$\Rightarrow \quad m_1 = 6, n_1 = 5$

So, $2m_1 + 3n_1 + m_1 n_1$

$= 2 \times 6 + 3 \times 5 + 6 \times 5 = 57$

6. **(6.00)** $f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$\Rightarrow \quad f_2'(x) = 2 \times 49 \times 50(x-1)^{49} - 50 \times 12 \times 49(x-1)^{48}$

$= 50 \times 49 \times 2(x-1)^{48}(x-1-6)$

$= 50 \times 49 \times 2(x-1)^{48}(x-7)$

$f_2(x)$ has local minimum at

$x = 7$ and no local maxima

$\Rightarrow \quad m_2 = 1, n_2 = 0$

$6m_2 + 4n_2 + 8m_2 n_2$

$= 6 \times 1 + 4 \times 0 + 8 \times 1 = 6$

7. **(1)** $y_n = \left(\frac{n+1}{n} \cdot \frac{n+2}{n} \cdot \frac{n+3}{n} \ldots\ldots \frac{n+n}{n}\right)^{1/n}$

$$\Rightarrow \log y_n = \frac{1}{n} \sum_{r=0}^{n} \log\left(1 + \frac{r}{n}\right)$$

$$\Rightarrow \left(\lim_{n\to\infty} \log y_n\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n} \log\left(1 + \frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \log(1+x)\,dx = [x \log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$= \log 2 - [x - \log|1+x|]_0^1$$

$$= \log 2 - 1 + \log 2 = 2\log 2 - 1$$

$$= \log 4 - \log e = \log\left(\frac{4}{e}\right)$$

$$\therefore \quad L = \frac{4}{e} \Rightarrow [L] = \left[\frac{4}{e}\right] = 1$$

8. **(7)** $\lim_{x\to 1} \frac{F(x)}{G(x)} = \frac{1}{14} \Rightarrow \lim_{x\to 1} \frac{\int_{-1}^x f(t)\,dt}{\int_{-1}^x t|f(f(t))|\,dt}$

$\because \int_{-1}^1 f(t)\,dt = 0$ and $\int_{-1}^1 t|f(f(t))|\,dt = 0$

$f(t)$ being odd function

$\therefore$ Using L Hospital's rule, we get

$$\lim_{x\to 1} \frac{f(x)}{x|f(f(x))|} = \frac{1}{14}$$

$$\Rightarrow \frac{f(1)}{|f(f(1))|} = \frac{1}{14} \Rightarrow \frac{1/2}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$$

$$\Rightarrow \left|f\left(\frac{1}{2}\right)\right| = 7 \Rightarrow f\left(\frac{1}{2}\right) = 7$$

9. **(a, d)**

$$\lim_{n\to\infty} \frac{\sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2} + \frac{1}{(an+2)^2} + \ldots + \frac{1}{(an+n)^2}\right)} = 54$$

$$\Rightarrow \frac{\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^{1/3}}{\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{\left(a + \frac{r}{n}\right)^2}} = 54$$

$$\Rightarrow \frac{\int_0^1 x^{1/3}\,dx}{\int_0^1 \frac{1}{(a+x)^2}\,dx} = 54 \Rightarrow \frac{\frac{3}{4}}{\frac{1}{a} - \frac{1}{a+1}} = 54$$

$\Rightarrow a^2 + a - 72 = 0 \Rightarrow (a+9)(a-8) = 0 \qquad \Rightarrow a = 8$ or -9

$\therefore$ options (a) and (d) are correct.

10. **(b, c)**

$$f(x) = \lim_{n\to\infty} \left[\frac{n^n (x+n)\left(x + \frac{n}{2}\right) \ldots \left(x + \frac{n}{n}\right)}{n!\left(x^2 + n^2\right)\left(x^2 + \frac{n^2}{4}\right) \ldots \left(x^2 + \frac{n^2}{n^2}\right)}\right]^{x/n}$$

$$= \lim_{n\to\infty} \left[\frac{\left(\frac{x}{n} + 1\right)\left(\frac{x}{n} + \frac{1}{2}\right) \ldots \left(\frac{x}{n} + \frac{1}{n}\right)}{\left(1.\frac{x^2}{n^2} + 1\right)\left(2.\frac{x^2}{n^2} + \frac{1}{2}\right) \ldots \left(n.\frac{x^2}{n^2} + \frac{1}{n}\right)}\right]^{x/n}$$

$$\Rightarrow \ln f(x)=\lim_{n\to\infty}\frac{x}{n}\left[\sum_{r=1}^{n}\ln\left(\frac{x}{n}+\frac{1}{r}\right)-\sum_{r=1}^{n}\ln\left(\frac{rx^2}{n^2}+\frac{1}{r}\right)\right]$$

$$=\lim_{n\to\infty}\frac{x}{n}\left[\sum_{r=1}^{n}\ln\left(1+\frac{rx}{n}\right)-\ln\left(1+\frac{r^2x^2}{n^2}\right)\right]$$

$$=x\int_0^1\ln(1+xy)\,dy-x\int_0^1\ln\left(1+x^2y^2\right)dy$$

Let $xy=t\Rightarrow x\,dy=dt$

$$\therefore\ \ln f(x)=\int_0^x\ln(1+t)\,dt-\int_0^x\ln\left(1+t^2\right)dt$$

$$\ln f(x)=\int_0^x\ln\left(\frac{1+t}{1+t^2}\right)dt\Rightarrow\frac{f'(x)}{f(x)}=\ln\left(\frac{1+x}{1+x^2}\right)$$

$$\Rightarrow\frac{f'(2)}{f(2)}=\ln\left(\frac{3}{5}\right)<0$$

$\Rightarrow f'(2)<0\ \therefore$ (c) is correct

and $\dfrac{f'(3)}{f(3)}=\ln\left(\dfrac{2}{5}\right)<\dfrac{f'(2)}{f(2)}$

$\therefore$ (d) is not correct

$\because f'(x)=f(x)\ \ln\left(\dfrac{1+x}{1+x^2}\right)>0,\ \forall\ x\in(0,1)$

$\Rightarrow f'(x)>0\ \forall x\in(0,1)$

$\therefore$ f is an increasing function.

$\therefore\ \dfrac{1}{2}<1\Rightarrow f\left(\dfrac{1}{2}\right)\le f(1)$

$\therefore$ (a) is not correct

and $\dfrac{1}{3}<\dfrac{2}{3}\Rightarrow f\left(\dfrac{1}{3}\right)\le f\left(\dfrac{2}{3}\right)$

$\therefore$ (b) is correct

11. **(a)** $g(a)=\lim\limits_{h\to0^+}\displaystyle\int_h^{1-h}t^{-a}(1-t)^{a-1}dt$

$$\therefore\ g\left(\frac{1}{2}\right)=\lim_{h\to0^+}\int_h^{1-h}t^{-1/2}(1-t)^{-1/2}dt$$

$$=\lim_{h\to0^+}\int_h^{1-h}\frac{1}{\sqrt{t(1-t)}}dt=\lim_{h\to0^+}\int_h^{1-h}\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(t-\frac{1}{2}\right)^2}}dt$$

$$=\lim_{h\to0^+}\left[\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_h^{1-h}=\lim_{h\to0^+}\left[\sin^{-1}(2t-1)\right]_h^{1-h}$$

$$=\lim_{h\to0^+}\left[\sin^{-1}(1-2h)-\sin^{-1}(2h-1)\right]$$

$$=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$$

12. **(d)** $g(a)=\lim\limits_{h\to0^+}\displaystyle\int_h^{1-h}t^{-a}(1-t)^{a-1}dt$

$$g(a)=\lim_{h\to0^+}\int_h^{1-h}(1-t)^{-a}t^{a-1}dt$$

$$\left(\text{using}\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right)$$

Also $g(1-a)=\lim\limits_{h\to0^+}\displaystyle\int_h^{1-h}t^{a-1}(1-t)^{-a}dt$

Thus $g(a)=g(1-a)$

$\Rightarrow g'(a)=-g'(1-a)\Rightarrow g'(a)+g'(1-a)=0$

Putting $a=\dfrac{1}{2}$ we get $g'\left(\dfrac{1}{2}\right)+g'\left(\dfrac{1}{2}\right)=0$

or $g'\left(\dfrac{1}{2}\right)=0$

13. Given $I_m=\displaystyle\int_0^{\pi}\frac{1-\cos mx}{1-\cos x}dx$

To prove: $I_m=m\pi,\ m=0,1,2,\ldots\ldots$

For m = 0

$$I_0=\int_0^{\pi}\frac{1-\cos 0}{1-\cos x}dx=\int_0^{\pi}\frac{1-1}{1-\cos x}dx=0$$

$\because$ Result is true for m = 0

For m = 1,

$$I_1=\int_0^{\pi}\frac{1-\cos x}{1-\cos x}dx=\int_0^{\pi}1.dx$$

$$(x)_0^{\pi}=\pi-0=\pi$$

$\therefore$ Result is true for $m=1$

Let the result be true for $m\le k$ i.e. $I_k=k\pi$ ....(i)

Consider $I_{k+1}=\displaystyle\int_0^{\pi}\frac{1-\cos(k+1)x}{1-\cos x}dx$

Now, $1-\cos(k+1)x$

$=1-\cos kx\cos x+\sin kx\sin x$

$=1+\cos kx\cos x+\sin kx\sin x-2\cos kx\cos x$

$=1+\cos(k-1)x-2\cos kx\cos x$

$=2-(1-\cos(k-1)x)-2\cos kx\cos x$

$=2-2\cos kx+2\cos kx-2\cos kx\cos x-[1-\cos(k-1)x]$

$=2(1-\cos kx)+2\cos kx(1-\cos x)-(1-\cos(k-1)x)$

$\therefore\ I_{k+1}=\int_0^{\pi}\frac{2(1-\cos kx)+2\cos kx(1-\cos x)-(1-\cos(k-1)x)}{1-\cos x}dx$

$=2\int_0^{\pi}\frac{1-\cos kx}{1-\cos x}dx+2\int_0^{\pi}\cos kx dx-\int_0^{\pi}\frac{1-\cos(k-1)x}{1-\cos x}dx$

$=2I_k+2\left(\frac{\sin kx}{k}\right)_0^{\pi}-I_{k-1}$

$=2(k\pi)+2(0)-(k-1)\pi$ [Using (i)]

$=(k+1)\pi$

Thus result is true for $m=k+1$ as well. Therefore by the principle of mathematical induction, given statement is true for all m = 0, 1, 2, ...

14. To prove that $\int_0^{n\pi+v}|\sin x|\,dx=2n+1-\cos v$

Let $I=\int_0^{n\pi+v}|\sin x|\,dx$

$=\int_0^{v}|\sin x|\,dx+\int_v^{n\pi+v}|\sin x|\,dx$

Now we know that $|\sin x|$ is a periodic function of period π, So using the property.

$=\int_a^{a+nT}f(x)dx=n\int_0^{T}f(x)dx$

where $n\in I$ and $f(x)$ is a periodic function of period T

We get, $I=\int_0^{v}\sin x\,dx+n\int_0^{\pi}\sin x\,dx$

[$\therefore$ $|\sin x|=\sin x$ for $0\le x\le v$]

$=(-\cos x)_0^{v}+n(-\cos x)_0^{\pi}=-\cos v+1+n(1+1)$

$=2n+1-\cos v=$ R.H.S.

15. We know that in integration as a limit sum

$$\int_0^1 f(x)dx=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}f(r/n)$$

Similarly the given series can be written as

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{6n}\right)=\lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+\frac{r}{n}}=\int_0^5\frac{1}{1+x}dx=[\log|1+x|]_0^5=\log 6$$